

GEOMETRIC AND HOMOLOGICAL PROPERTIES OF AFFINE DELIGNE-LUSZTIG VARIETIES

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ABSTRACT. This paper studies affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ in the affine flag variety of a tamely ramified group. We describe the geometric structure of $X_{\tilde{w}}(b)$ for \tilde{w} minimal length element in a conjugacy class of extended affine Weyl group, generalizing one of the main results in [18]. We then provide a reduction method relating $X_{\tilde{w}}(b)$ for arbitrary \tilde{w} in the extended affine Weyl group to those associated to a minimal length element. Based on this, we establish a connection between the dimension of affine Deligne-Lusztig variety and the degree of class polynomial of affine Hecke algebra and as a consequence, prove a conjecture of Görtz, Haines, Kottwitz and Reuman in [10].

INTRODUCTION

0.1. The main purpose of this paper is to discuss some geometric and homological properties of affine Deligne-Lusztig varieties in the affine flag variety of a tamely ramified group.

To provide some context, we begin with (finite) Deligne-Lusztig varieties. Let G be a connected reductive algebraic group over an algebraic closure \mathbf{k} of a finite field \mathbb{F}_q and B be a Borel subgroup. Let W be the Weyl group. Then we have the Bruhat decomposition $G = \sqcup_{w \in W} B\dot{w}B$. Here $\dot{w} \in G$ is a representative of $w \in W$.

Let σ be the Frobenius automorphism on G . Following [6], the Deligne-Lusztig variety associated to $w \in W$ is a locally closed subvariety of the flag variety G/B defined by

$$X_w = \{gB \in G/B; g^{-1}\sigma(g) \in B\dot{w}B\}.$$

It is known that X_w is always nonempty and is a smooth variety of dimension $\ell(w)$.

The finite group G^σ acts on X_w and hence on its cohomology. Deligne and Lusztig showed in [6] that every irreducible representations of G^σ can be realized as a direct summand of (l -adic) cohomology with compact support of some Deligne-Lusztig variety, with coefficients in a certain local system. In [30], Lusztig used these cohomology to classify the irreducible representations of G^σ .

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0.2. The term “affine Deligne-Lusztig variety” was first introduced by Rapoport in [33]. Here “affine” refers to the fact that the notion is defined in terms of affine root systems, which arises from loop groups.

For simplicity, we restrict to split case here. Let G be a connected reductive group split over \mathbb{F}_q and $L = \mathbf{k}((\epsilon))$ be the field of Laurent series. The Frobenius automorphism σ on G induces an automorphism on the loop group $G(L)$, which we still denote by the same symbol.

Let I be an Iwahori subgroup of $G(L)$. By definition, the affine Deligne-Lusztig variety associated with \tilde{w} in the extended affine Weyl group $\tilde{W} \cong I \backslash G(L) / I$ and $b \in G(L)$ is

$$X_{\tilde{w}}(b) = \{gI \in G(L)/I; g^{-1}b\sigma(g) \in I\tilde{w}I\}.$$

Here $\tilde{w} \in G(L)$ is a representative of $\tilde{w} \in \tilde{W}$.

Understanding the emptiness/nonemptiness pattern and dimension of affine Deligne-Lusztig variety is fundamental for understanding certain aspects of Shimura varieties with Iwahori level structure. On the special fiber of Shimura variety, we have two important stratifications: one is the Newton stratification whose strata are indexed by certain σ -conjugacy classes $[b] \subseteq G(L)$; the other one is the Kottwitz-Rapoport stratification whose strata are indexed by certain elements \tilde{w} in the extended affine Weyl group \tilde{W} . There is a close relation between the affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ and the intersection of the Newton stratum associated with $[b]$ and the Kottwitz-Rapoport stratum associated with \tilde{w} (see [13] and [38]). A joint work with Wedhorn [20] shows that affine Deligne-Lusztig variety is also fundamental to the study of parahoric reduction of Shimura varieties.

There is also an important connection between affine Deligne-Lusztig varieties and moduli spaces of p -divisible groups and their analogy in the function field case, local G -shtukas. See [23].

0.3. The affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ for arbitrary \tilde{w} and b is very difficult to understand. One of the main goal of this paper is to develop a reduction method to study geometric and homological properties on $X_{\tilde{w}}(b)$.

The reduction method is a combination of combinatorial, of algorithmic, of geometric and of representation-theoretic method. To explain why and how it works, let us first discuss the reason why affine Deligne-Lusztig variety is more complicated than its finite counterpart.

In finite case, Lang’s theorem implies that G is a single σ -conjugacy class. This is the reason that Deligne-Lusztig variety only depends on the parameter $w \in W$ and there is no need to choose an element $b \in G$. However, in the affine setting, Lang’s theorem fails. Therefore affine Deligne-Lusztig variety depends on two parameters: \tilde{w} in the extended affine Weyl group and an element b (or its σ -conjugacy class) in the loop group. Hence even describing when $X_{\tilde{w}}(b)$ is nonempty is a challenging task.

0.4. To overcome the difficulty, we prove that Lang’s theorem holds “locally” for loop group. This is obtained via a reduction method as follows.

We start with Iwahori-Bruhat decomposition $G(L) = \sqcup_{\tilde{w} \in \tilde{W}} I\tilde{w}I$. Then $G(L) = \cup_{\tilde{w} \in \tilde{W}} G(L) \cdot_{\sigma} I\tilde{w}I$. Here \cdot_{σ} means σ -twisted conjugation action.

Our strategy is outlined as follows.

$$\begin{array}{ccc} \{G(L) \cdot_{\sigma} I\tilde{w}I, \forall \tilde{w} \in \tilde{W}\} & \xrightarrow{(1)} & \{G(L) \cdot_{\sigma} I\tilde{w}I, \forall \tilde{w} \in \tilde{W}_{\min}\} \\ & \xleftarrow{(2)} & \\ \{G(L) \cdot_{\sigma} I\tilde{w}I, \forall \tilde{w} \text{ straight}\} & \xrightarrow{(3)} & \{G(L) \cdot_{\sigma} \tilde{w}, \forall \tilde{w} \text{ straight}\} \end{array}$$

Here \tilde{W}_{\min} is the set of elements in \tilde{W} that are of minimal length in their conjugacy classes.

In step (1), we apply a variation of “reduction method” à la Deligne and Lusztig [6]. A recent joint work with Nie [19] showed that minimal length element satisfies special property which allows us to reduce arbitrary element to some minimal length element.

An element \tilde{w} in \tilde{W} is called straight if $\ell(\tilde{w}^n) = n\ell(\tilde{w})$ for all $n \in \mathbb{N}$. A conjugacy class contains a straight element is called a straight conjugacy class. By [19], a minimal length element differs from a straight element by an element in a finite Coxeter group. Thus we may apply Lang’s theorem and reduce further to straight elements. This is step (2).

Moreover, any straight element can be regarded as a basic element in the extended affine Weyl group of some Levi subgroup of G . Thus using P -alcove introduced in [10] and its generalization in [11], one can show that any element in $I\tilde{w}I$ is σ -conjugate to \tilde{w} for straight \tilde{w} . This finishes step (3) and the reduction.

Combined with a disjointness result (Proposition 3.4), we obtained a natural bijection between straight conjugacy classes of the extended affine Weyl group \tilde{W} and σ -conjugacy classes of the loop group $G(L)$. This gives a new proof of Kottwitz’s classification of σ -conjugacy classes [25] and [26].

0.5. Similar strategy applies to the study of affine Deligne-Lusztig varieties and reduce the study of many geometric and homological properties on arbitrary affine Deligne-Lusztig variety to those associated to minimal length elements.

Although the structure for arbitrary affine Deligne-Lusztig variety is quite complicated, the minimal length ones have very nice geometric structure. In Theorem 4.7, we show that $X_{\tilde{w}}(b)$, for \tilde{w} a minimal length element in its conjugacy class in \tilde{W} , is nonempty if and only if b is σ -conjugate to \tilde{w} and in this case, $X_{\tilde{w}}(b)$ is a union of (possibly

infinitely many) copies of finite Deligne-Lusztig variety associated to some reductive group and the orbit $\mathbb{J}_b \backslash X_{\tilde{w}}(b)$ under the action of the σ -centralizer \mathbb{J}_b of b is in a natural bijection with the orbit space of an affine space by an action of a finite torus. This generalizes one of the main theorem in [18].

As a consequence, reduction method works for the homology of affine Deligne-Lusztig variety and we prove in section 5 that for superstraight element $x \in \tilde{W}$, the action of σ -centralizer \mathbb{J}_x on the Borel-Moore homology of affine Deligne-Lusztig variety $X_{\tilde{w}}(x)$ factors through the action of the centralizer of x in the extended affine Weyl group W . The definition of superstraight element can be found in §5.2, which includes generic elements in the maximal torus and superbasic element for type A as special cases.

0.6. Emptiness/nonemptiness pattern and dimension formula for affine Deligne-Lusztig variety is obtained if one can keep track of the reductive step from an arbitrary element to a minimal length element. This is achieved via the class polynomial of affine Deligne-Lusztig variety. In section 6, we prove the “dimension=degree” theorem, which provides a dictionary between affine Deligne-Lusztig variety and affine Hecke algebra in the following sense:

- (1) The affine Deligne-Lusztig variety is nonempty if and only if certain class polynomial of affine Hecke algebra is nonzero.
- (2) The dimension of affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ equals $\frac{1}{2}\ell(\tilde{w})$ minus the length of the newton point of b , plus some correction term given by the degree of corresponding class polynomials.
- (3) In the split case, if b is superbasic, then the number of rational points in the affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ can be calculated by corresponding class polynomial. See section 8.

As a consequence, in Theorem 7.1 we solve the emptiness/nonemptiness question for affine Deligne-Lusztig variety in affine Grassmannian for tamely ramified groups, generalizing previous results of Rapoport-Richartz [34], Kottwitz-Rapoport [28], Lucarelli [31] and Gashi [7] for unramified groups.

0.7. In the remaining sections (section 9-section 11), we study $X_{\tilde{w}}(b)$ for b basic and \tilde{w} in the lowest two sided cell of \tilde{W} and verify a main conjecture of Görtz-Haines-Kottwitz-Reuman [10] and its generalization. This is achieved by combining the “dimension=degree” theorem with the partial conjugation method developed in [14] and the dimension formula for affine Deligne-Lusztig variety in affine Grassmannian [9] and [37]

For arbitrary \tilde{w} and b , we also give an upper bound for the dimension of $X_{\tilde{w}}(b)$.

1. GROUP THEORETICAL DATA

1.1. Let \mathbf{k} be an algebraic closure of a finite field \mathbb{F}_q . Let $F = \mathbb{F}_q((\epsilon))$ and $L = \mathbf{k}((\epsilon))$ be the fields of Laurent series. Let G be a connected reductive group over F and splits over a tamely ramified extension of L .

Let $S \subseteq G$ be a maximal L -split torus defined over F and $T = \mathcal{Z}_G(S)$ be its centralizer. Since G is quasi-split over L , T is a maximal torus. Let N be the normalizer of T . By definition, the *finite Weyl group associated to S* is

$$W = N(L)/T(L)$$

and the *Iwahori-Weyl group associated to S* is

$$\tilde{W} = N(L)/T(L)_1,$$

Here $T(L)_1$ denotes the unique parahoric subgroup of $T(L)$.

Let σ be the Frobenius automorphism in $\text{Gal}(L/F)$, defined by $\sigma(\sum a_n \epsilon^n) = \sum a_n^q \epsilon^n$. We also denote the induced automorphism on $G(L)$ by σ . Then σ acts on W and \tilde{W} in a natural way. We denote the corresponding automorphism by δ .

We denote by \mathcal{A} the apartment of G_L corresponding to S . We fix a σ -invariant alcove \mathfrak{a}_C in \mathcal{A} . We denote by $I \subseteq G(L)$ the Iwahori subgroup corresponding to \mathfrak{a}_C over L and by $\tilde{\mathbb{S}}$ the set of simple reflections at the walls of \mathfrak{a}_C . Let $G_1 \subseteq G(L)$ be the subgroup generated by all parahoric subgroups. Set $N_1 = N(L) \cap G_1$. By [5, Proposition 5.2.12], the quadruple $(G_1, I, N_1, \tilde{\mathbb{S}})$ is a double Tits system with affine Weyl group

$$W_a = (N(L) \cap G_1)/(N(L) \cap I).$$

In particular,

$$G(L) = \sqcup_{\tilde{w} \in \tilde{W}} I \dot{\tilde{w}} I, \quad \text{and } G(L)_1 = \sqcup_{\tilde{w} \in W_a} I \dot{\tilde{w}} I.$$

Here $\dot{\tilde{w}}$ is a representative in $N(L)$ of the element $\tilde{w} \in \tilde{W}$.

1.2. Let Γ be the absolute Galois group $\text{Gal}(L^{sep}/L)$, here L^{sep} is a separable closure of L . Let P be the Γ -coinvariants of $X_*(T)$. By [32, Section 5], $P \cong T(L)/T(L)_1$ and by [22, Proposition 13],

$$\tilde{W} = P \rtimes W = \{t^\mu w; \mu \in P, w \in W\}.$$

Let Φ be the set of (relative) roots of G over L with respect to S and Φ_a the set of affine roots. Let $\mathbb{S} \subseteq \Phi$ be the set of simple roots. We identify \mathbb{S} with the set of simple reflections in W . Then $\mathbb{S} \subseteq \tilde{\mathbb{S}}$. Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots of Φ .

By [22], there exists a reduced root system Σ such that

$$W_a = Q^\vee(\Sigma) \rtimes W(\Sigma),$$

where $Q^\vee(\Sigma)$ is the coroot lattice of Σ . In other words, we identify $Q^\vee(\Sigma)$ with $X_*(T_{sc})_\Gamma$ and $W(\Sigma)$ with W . We simply write Q for $Q^\vee(\Sigma)$.

For $a \in \Phi$, we denote by $U_a \subseteq G$ the corresponding root subgroup and for $\alpha \in \Phi_a$, we denote by $U'_\alpha \subseteq G(L)$ the corresponding root subgroup scheme over \mathbf{k} . By [32, section 9], any affine root is of the form $a+m$ for a finite root $a \in \Phi$ and $m \in \mathbb{Q}$ and U'_α is one-dimensional for all affine root α . Moreover, if $2a \in \Phi$, then $m \in \mathbb{Z}$; if $\frac{1}{2}a \in \Phi$, then $m \in \frac{1}{2} + \mathbb{Z}$. We call a root $a \in \Phi$ a finite simple root if $-a$ (in Φ_a) is a simple affine root with respect to I . Our convention here is consistent with [10].

For any element $\tilde{w} \in \tilde{W}$, the length $\ell(\tilde{w})$ is the number of “affine root hyperplanes” in \mathcal{A} separating $\tilde{w}(\mathfrak{a}_C)$ from \mathfrak{a}_C .

Let Ω be the subgroup of \tilde{W} consisting of length 0 elements. Then Ω is the stabilizer of the base alcove \mathfrak{a}_C in \tilde{W} . Let $\tau \in \Omega$, then for any $w, w' \in W_a$, we say that $\tau w \leq \tau w'$ if $w \leq w'$ for the Bruhat order on W_a .

For any $J \subseteq \tilde{\mathbb{S}}$, let W_J be the subgroup of \tilde{W} generated by s_j for $j \in J$ and ${}^J\tilde{W}$ (resp. \tilde{W}^J) be the set of minimal elements for the cosets $W_J \backslash \tilde{W}$ (resp. \tilde{W}/W_J). For $J, J' \subseteq \tilde{\mathbb{S}}$, we simply write ${}^J\tilde{W}^{J'}$ for ${}^J\tilde{W} \cap \tilde{W}^{J'}$. If W_J is finite, then we denote by w_0^J its longest element in W_J . We simply write w_0 for $w_0^{\tilde{\mathbb{S}}}$.

For $J \subseteq \mathbb{S}$, let Φ_J be the set of roots that are linear combinations of $(\alpha_j)_{j \in J}$. Let $\Phi_J^+ = \Phi_J \cap \Phi^+$ and $\Phi_J^- = \Phi_J \cap \Phi^-$. Let $M_J \subseteq G$ be the subgroup generated by T and U_a for $a \in \Phi_J$. Then $\tilde{W}_J = P \rtimes W_J$ is the Iwahori-Weyl group of M_J .

1.3. Two elements \tilde{w}, \tilde{w}' of \tilde{W} are said to be δ -conjugate if $\tilde{w}' = \tilde{x}\tilde{w}\delta(\tilde{x})^{-1}$ for some $\tilde{x} \in \tilde{W}$. The relation of δ -conjugacy is an equivalence relation and the equivalence classes are said to be δ -conjugacy classes.

Let $(P/Q)_\delta$ be the δ -coinvariants on P/Q . Let

$$\kappa : \tilde{W} \rightarrow \tilde{W}/W_a \cong P/Q \rightarrow (P/Q)_\delta$$

be the natural projection. We call κ the *Kottwitz map*.

Let $P_{\mathbb{Q}} = P \otimes_{\mathbb{Z}} \mathbb{Q}$ and $P_{\mathbb{Q}}/W$ the quotient of $P_{\mathbb{Q}}$ be the natural action of W . We may identify $P_{\mathbb{Q}}/W$ with $P_{\mathbb{Q},+}$, where

$$P_{\mathbb{Q},+} = \{\chi \in P_{\mathbb{Q}}; \alpha(\chi) \geq 0, \text{ for all } \alpha \in \Phi^+\}.$$

Let $P_{\mathbb{Q},+}^\delta$ be its δ -invariants.

Now we define a map from \tilde{W} to $P_{\mathbb{Q},+}^\delta$ as follow. For each element $\tilde{w} = t^x w \in \tilde{W}$, there exists $n \in \mathbb{N}$ such that $\delta^n = 1$ and $w\delta(w)\delta^2(w) \cdots \delta^{n-1}(w) = 1$. This is because $W \rtimes \langle \delta \rangle \subseteq \text{Aut}(W)$ is a finite group. Then $\tilde{w}\delta(\tilde{w}) \cdots \delta^{n-1}(\tilde{w}) = t^\lambda$ for some $\lambda \in P$. Let $\nu_{\tilde{w}} = \lambda/n \in P_{\mathbb{Q}}$ and $\bar{\nu}_{\tilde{w}}$ the corresponding element in $P_{\mathbb{Q},+}$. It is easy to see that $\nu_{\tilde{w}}$ is independent of the choice of n .

Let $\bar{\nu}_{\tilde{w}}$ be the unique element in $P_{\mathbb{Q},+}$ that lies in the W -orbit of $\nu_{\tilde{w}}$. Since $t^\lambda = \tilde{w}t^{\delta(\lambda)}\tilde{w}^{-1} = t^{w\delta(\lambda)}$, $\bar{\nu}_{\tilde{w}} \in P_{\mathbb{Q},+}^\delta$. We call the map $\tilde{W} \rightarrow P_{\mathbb{Q},+}^\delta$, $\tilde{w} \mapsto \bar{\nu}_{\tilde{w}}$ the *Newton map*.

Define $f : \tilde{W} \rightarrow P_{\mathbb{Q},+}^\delta \times (P/Q)_\delta$ by $\tilde{w} \mapsto (\bar{\nu}_{\tilde{w}}, \kappa(\tilde{w}))$. It is the restriction to \tilde{W} of the map $G(L) \rightarrow P_{\mathbb{Q},+}^\delta \times (P/Q)_\delta$ in [26, 4.13] and is constant on δ -each conjugacy class of \tilde{W} . We denote the image of the map f by $B(\tilde{W}, \delta)$.

2. SOME SPECIAL PROPERTIES OF AFFINE WEYL GROUPS

In this section, we recollect some special properties on affine Weyl group established in a joint work with Nie [19]. These properties will play a crucial role in the rest of this paper.

2.1. For $\tilde{w}, \tilde{w}' \in \tilde{W}$ and $i \in \tilde{S}$, we write $\tilde{w} \xrightarrow{s_i}_\delta \tilde{w}'$ if $\tilde{w}' = s_i \tilde{w} s_{\delta(i)}$ and $\ell(\tilde{w}') \leq \ell(\tilde{w})$. We write $\tilde{w} \xrightarrow{\sim}_\delta \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n = \tilde{w}'$ of elements in \tilde{W} such that for any k , $\tilde{w}_k = \tau \tilde{w}_{k-1} \delta(\tau)^{-1}$ for some $\tau \in \Omega$ or $\tilde{w}_{k-1} \xrightarrow{s_i}_\delta \tilde{w}_k$ for some $i \in \tilde{S}$. We write $\tilde{w} \approx_\delta \tilde{w}'$ if $\tilde{w} \xrightarrow{\sim}_\delta \tilde{w}'$ and $\tilde{w}' \xrightarrow{\sim}_\delta \tilde{w}$. It is easy to see that $\tilde{w} \approx_\delta \tilde{w}'$ if $\tilde{w} \xrightarrow{\sim}_\delta \tilde{w}'$ and $\ell(\tilde{w}) = \ell(\tilde{w}')$.

We call $\tilde{w}, \tilde{w}' \in \tilde{W}$ *elementarily strongly δ -conjugate* if $\ell(\tilde{w}) = \ell(\tilde{w}')$ and there exists $\tilde{x} \in \tilde{W}$ such that $\tilde{w}' = \tilde{x} \tilde{w} \delta(\tilde{x})^{-1}$ and $\ell(\tilde{x} \tilde{w}) = \ell(\tilde{x}) + \ell(\tilde{w})$ or $\ell(\tilde{w} \delta(\tilde{x})^{-1}) = \ell(\tilde{x}) + \ell(\tilde{w})$. We call \tilde{w}, \tilde{w}' *strongly δ -conjugate* if there is a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n = \tilde{w}'$ such that for each i , \tilde{w}_{i-1} is elementarily strongly δ -conjugate to \tilde{w}_i . We write $\tilde{w} \sim_\delta \tilde{w}'$ if \tilde{w} and \tilde{w}' are strongly δ -conjugate.

The following result is proved in [19, Theorem 2.10] and is a key ingredient in the reductive step (1) in §0.4.

Theorem 2.1. *Let \mathcal{O} be a δ -conjugacy class in \tilde{W} and \mathcal{O}_{\min} be the set of minimal length elements in \mathcal{O} . Then*

(1) *For each element $\tilde{w} \in \mathcal{O}$, there exists $\tilde{w}' \in \mathcal{O}_{\min}$ such that $\tilde{w} \rightarrow_\delta \tilde{w}'$.*

(2) *Let $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$, then $\tilde{w} \sim_\delta \tilde{w}'$.*

2.2. Let \tilde{H} be the Hecke algebra associated to \tilde{W} , i.e., \tilde{H} is the associated $A = \mathbb{Z}[v, v^{-1}]$ -algebra with basis $T_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}$ and multiplication is given by

$$\begin{aligned} T_{\tilde{x}} T_{\tilde{y}} &= T_{\tilde{x}\tilde{y}}, & \text{if } l(\tilde{x}) + l(\tilde{y}) = l(\tilde{x}\tilde{y}); \\ (T_s - v)(T_s + v^{-1}) &= 0, & \text{for } s \in \tilde{S}. \end{aligned}$$

Then $T_s^{-1} = T_s - (v - v^{-1})$ and $T_{\tilde{w}}$ is invertible in \tilde{H} for all $\tilde{w} \in \tilde{W}$.

The map $T_{\tilde{w}} \mapsto T_{\delta(w)}$ defines an A -algebra automorphism of \tilde{H} , which we still denote by δ .

Let $h, h' \in \tilde{H}$, we call $[h, h']_\delta = hh' - h'\delta(h)$ the δ -commutator of h and h' . Let $[\tilde{H}, \tilde{H}]_\delta$ the A -submodule of \tilde{H} generated by all δ -commutators.

By Theorem 2.1, for any δ -conjugacy class \mathcal{O} of \tilde{W} and $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$, we have that $T_{\tilde{w}} \equiv T_{\tilde{w}'} \pmod{[\tilde{H}, \tilde{H}]_\delta}$. See [19, Lemma 5.1].

Now for any δ -conjugacy class \mathcal{O} , we fix a minimal length representative $\tilde{w}_\mathcal{O}$. Then the image of $T_{\tilde{w}_\mathcal{O}}$ in $\tilde{H}/[\tilde{H}, \tilde{H}]_\delta$ is independent of the choice of $\tilde{w}_\mathcal{O}$. It is proved in [19, Theorem 6.7] that

Theorem 2.2. *The elements $T_{\tilde{w}_\mathcal{O}}$ forms a A -basis of $\tilde{H}/[\tilde{H}, \tilde{H}]_\delta$, here \mathcal{O} runs over all the δ -conjugacy class of \tilde{W} .*

2.3. Now for any $\tilde{w} \in \tilde{W}$ and a δ -conjugacy class \mathcal{O} , there exists unique $f_{\tilde{w}, \mathcal{O}} \in A$ such that

$$T_{\tilde{w}} \equiv \sum_{\mathcal{O}} f_{\tilde{w}, \mathcal{O}} T_{\tilde{w}_\mathcal{O}} \pmod{[\tilde{H}, \tilde{H}]_\delta}.$$

By [19, Theorem 5.3], $f_{\tilde{w}, \mathcal{O}}$ is a polynomial in $\mathbb{Z}[v - v^{-1}]$ with non-negative coefficient and can be constructed inductively as follows.

If \tilde{w} is a minimal element in a δ -conjugacy class of \tilde{W} , then we set

$$f_{\tilde{w}, \mathcal{O}} = \begin{cases} 1, & \text{if } \tilde{w} \in \mathcal{O} \\ 0, & \text{if } \tilde{w} \notin \mathcal{O} \end{cases}.$$

Now we assume that \tilde{w} is not a minimal element in the conjugacy class of \tilde{W} that contains it and that for any $\tilde{w}' \in \tilde{W}$ with $\ell(\tilde{w}') < \ell(\tilde{w})$, $f_{\tilde{w}, \mathcal{O}}$ is constructed. By our assumption on (\tilde{W}, δ) , there exists $\tilde{w}_1 \approx_\delta \tilde{w}$ and $i \in \tilde{S}$ such that $\ell(s_i \tilde{w}_1 s_{\delta(i)}) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. In this case, $\ell(s_i \tilde{w}) < \ell(\tilde{w})$ and we define $f_{\tilde{w}, \mathcal{O}}$ as

$$f_{\tilde{w}, \mathcal{O}} = (v - v^{-1}) f_{s_i \tilde{w}_1, \mathcal{O}} + f_{s_i \tilde{w}_1 s_{\delta(i)}, \mathcal{O}}.$$

2.4. We call an element $\tilde{w} \in \tilde{W}$ a δ -straight element if $\ell(\tilde{w}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$. By [17, Lemma 1.1], \tilde{w} is δ -straight if and only if for any $m \in \mathbb{N}$, $\ell(\tilde{w}\delta(\tilde{w}) \cdots \delta^{m-1}(\tilde{w})) = m\ell(\tilde{w})$. We call a δ -conjugacy class *straight* if it contains some straight element. As we'll see later, there is a natural bijection between the set of straight δ -conjugacy classes of \tilde{W} and the set of σ -conjugacy classes of loop group.

We have the following results on straight elements and straight conjugacy classes.

(1) The map $f : \tilde{W} \rightarrow P_{\mathbb{Q},+}^\delta \times (P/Q)_\delta$ induces a bijection from the set of straight δ -conjugacy classes to $B(\tilde{W}, \delta)$. See [19, Theorem 3.3].

(2) Let \mathcal{O} be a straight δ -conjugacy class of \tilde{W} and $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$. Then $\tilde{w} \approx_\delta \tilde{w}'$. See [19, Theorem 3.9].

The following result [19, Proposition 2.8] relates minimal length element with straight elements as needed in §0.4 step (2).

Theorem 2.3. *Let \mathcal{O} be a δ -conjugacy class of \tilde{W} and $\tilde{w} \in \mathcal{O}$. Then there exists $\tilde{w}' \in \mathcal{O}_{\min}$ such that*

- (1) $\tilde{w} \xrightarrow{\delta} \tilde{w}'$;
- (2) *There exists $J \subseteq \tilde{S}$ with W_J is finite, a straight element $x \in \tilde{W}$ with $x \in {}^J\tilde{W}^{\delta(J)}$ and $x\delta(J) = J$ and $u \in W_J$ such that $\tilde{w}' = ux$.*

Remark. In the setting of Theorem 2.3 (2), we have that $f(\tilde{w}) = f(\tilde{w}') = f(x)$. See [17, Proposition 1.2].

2.5. Let \mathcal{O} be a δ -conjugacy class. Set $\nu_{\mathcal{O}} = \bar{\nu}_{\tilde{w}}$ for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$ and $J_{\mathcal{O}} = \{i \in S; \langle \nu_{\mathcal{O}}, \alpha_i \rangle = 0\}$. The following description of straight conjugacy classes [19, Proposition 3.2] will be used in step (3) of §0.4.

Proposition 2.4. *Let \mathcal{O} be a δ -conjugacy class of \tilde{W} . Then \mathcal{O} is straight if and only if \mathcal{O} contains a length 0 element in \tilde{W}_J for some $J \subseteq S$ with $\delta(J) = J$. In this case, there exists a length 0-element x in $\tilde{W}_{J_{\mathcal{O}}}$ and $y \in W^{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $yx\delta(y)^{-1} \in \mathcal{O}_{\min}$.*

2.6. Any fiber of $f : \tilde{W} \rightarrow B(\tilde{W}, \delta)$ is a union of δ -conjugacy classes. As we'll see later, the case that the fiber is a single δ -conjugacy class is of particular interest. We call such δ -conjugacy class *superstraight*. A minimal length element in a superstraight δ -conjugacy class is called a *δ -superstraight element*.

Let $\tilde{w} \in \Omega$. Then the conjugation by \tilde{w} gives a permutation on the set of simple reflections \tilde{S} . We call that \tilde{w} is *δ -superbasic* if each orbit of $\tilde{w} \circ \delta$ on \tilde{S} is a union of connected components of the corresponding Dynkin diagram of \tilde{S} . By [19, §3.5], \tilde{w} is a δ -superbasic element if and only if $\tilde{W} = W_1^{m_1} \times \cdots \times W_l^{m_l}$, where W_i is an affine Weyl group of type \tilde{A}_{n_i-1} and $\tau \circ \delta$ gives an order $n_i m_i$ permutation on the set of simple reflections in $W_i^{m_i}$.

The following description of superstraight conjugacy class is obtained in [19, Proposition 3.4], which is analogous to Proposition 2.4.

Proposition 2.5. *Let \mathcal{O} be a δ -conjugacy class of \tilde{W} . Then \mathcal{O} is superstraight if and only if there exists a δ -superbasic element x in $\tilde{W}_{J_{\mathcal{O}}}$ and $y \in W^{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $yx\delta(y)^{-1} \in \mathcal{O}_{\min}$.*

Corollary 2.6. *Let \mathcal{O} be a δ -straight conjugacy class of \tilde{W} . If $\nu_{\mathcal{O}}$ is regular, then \mathcal{O} is superstraight.*

2.7. In particular, let δ be the identity map and $\lambda \in P_+$ be a regular element. Let $\mathcal{O} = \{t^{w\lambda}; w \in W\}$ be the conjugacy class of t^λ . Then $\nu_{\mathcal{O}} = \lambda$ is regular and \mathcal{O} is superstraight. Since every element in \mathcal{O} is of the same length, t^λ is a minimal length element in \mathcal{O} and hence is a superstraight element.

3. σ -CONJUGACY CLASSES

3.1. We consider σ -conjugation action on $G(L)$, $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$. The classification of σ -conjugacy classes is due to Kottwitz [25] and [26]. In order to understand affine Deligne-Lusztig varieties, we also need some information on the relation between σ -conjugacy classes and Iwahori-Bruhat cells $I\dot{w}I$. In this section, we'll study the subsets of the form

$$G(L) \cdot_\sigma I\dot{w}I = \{gg'\sigma(g)^{-1}; g \in G(L), g' \in I\dot{w}I\}$$

for any $\tilde{w} \in \tilde{W}$. As a consequence, we obtain a new proof of Kottwitz's classification.

Lemma 3.1. *Let $\tilde{w}, \tilde{w}' \in \tilde{W}$.*

(1) *If $\tilde{w} \xrightarrow{\delta} \tilde{w}'$, then*

$$G(L) \cdot_\sigma I\dot{w}'I \subseteq G(L) \cdot_\sigma I\dot{w}I \subseteq G(L) \cdot_\sigma I\dot{w}'I \cup \bigcup_{\tilde{x} \in \tilde{W}, \ell(\tilde{x}) < \ell(\tilde{w})} G(L) \cdot_\sigma I\dot{x}I.$$

(2) *If $\tilde{w} \approx_\delta \tilde{w}'$, then*

$$G(L) \cdot_\sigma I\dot{w}I = G(L) \cdot_\sigma I\dot{w}'I.$$

Proof. It suffices to consider the case where $\tilde{w}' = \tau\tilde{w}\delta(\tau)^{-1}$ for some $\tau \in \Omega$ or $\tilde{w} \xrightarrow{i} \tilde{w}'$ for some $i \in \tilde{S}$.

If $\tilde{w}' = \tau\tilde{w}\tau^{-1}$, then $I\dot{w}'I = \dot{\tau}I\dot{w}I\sigma(\dot{\tau})^{-1}$ and $G(L) \cdot_\sigma I\dot{w}I = G(L) \cdot_\sigma I\dot{w}'I$.

Now we consider the case where $\tilde{w}' = s_i\tilde{w}s_{\delta(i)}$ and $\ell(\tilde{w}') \leq \ell(\tilde{w})$. By [6, Lemma 1.6.4], we have that $\tilde{w} = \tilde{w}'$ or $s_i\tilde{w} < \tilde{w}$ or $\tilde{w}s_{\delta(i)} < \tilde{w}$. Now we prove the case where $s_i\tilde{w} < \tilde{w}$. The case $\tilde{w}s_{\delta(i)} < \tilde{w}$ can be proved in the same way.

Since $s_i\tilde{w} < \tilde{w}$, then

$$G(L) \cdot_\sigma I\dot{w}I = G(L) \cdot_\sigma I\dot{s}_i I\dot{s}_i \dot{w}I = G(L) \cdot_\sigma I\dot{s}_i \dot{w} \sigma(I\dot{s}_i I) = G(L) \cdot_\sigma I\dot{s}_i \dot{w} I\dot{s}_{\delta(i)} I.$$

Moreover,

$$I\dot{s}_i \dot{w} I\dot{s}_{\delta(i)} I = \begin{cases} I\dot{w}'I, & \text{if } \ell(\tilde{w}') = \ell(s_i\tilde{w}) + 1 = \ell(\tilde{w}); \\ I\dot{s}_i \dot{w} I \sqcup I\dot{w}'I, & \text{if } \ell(\tilde{w}') = \ell(s_i\tilde{w}) - 1 = \ell(\tilde{w}) - 2. \end{cases}$$

In either case,

$$G(L) \cdot_\sigma I\dot{w}'I \subseteq G(L) \cdot_\sigma I\dot{w}I \subseteq G(L) \cdot_\sigma I\dot{w}'I \cup \bigcup_{\tilde{x} \in \tilde{W}, \ell(\tilde{x}) < \ell(\tilde{w})} G(L) \cdot_\sigma I\dot{x}I.$$

If moreover, $\tilde{w} \approx_\delta \tilde{w}'$, then $\ell(\tilde{w}') = \ell(\tilde{w})$. In this case, $G(L) \cdot_\sigma I\dot{w}I = G(L) \cdot_\sigma I\dot{w}'I$. \square

Lemma 3.2. *Let $J \subseteq \tilde{S}$ with W_J finite and $x \in {}^J\tilde{W}^{\delta(J)}$ with $x\delta(J) = J$. Then for any $u \in W_J$, we have that*

$$G(L) \cdot_\sigma I\dot{u}I = G(L) \cdot_\sigma I\dot{x}I.$$

Proof. Let \mathcal{P} be the standard parahoric subgroup corresponding to J and \mathcal{U} be its prounipotent radical. Let $\bar{\mathcal{P}} = \mathcal{P}/\mathcal{U}$ be the reductive quotient of \mathcal{P} and $p \mapsto \bar{p}$ the projection from \mathcal{P} to $\bar{\mathcal{P}}$. By definition, $\bar{p} \mapsto \bar{x}\sigma(\bar{p})\bar{x}^{-1}$ is a Frobenius morphism on $\bar{\mathcal{P}}$. We denote it by $\sigma_{\bar{x}}$. Hence by Lang's theorem, $\bar{\mathcal{P}} = \{\bar{p}^{-1}\bar{x}\sigma(\bar{p})\bar{x}^{-1}; \bar{p} \in \bar{\mathcal{P}}\}$ and

$$\mathcal{P}\bar{x}/\sigma(\mathcal{U}) = \bar{x}\sigma(\mathcal{P})/\sigma(\mathcal{U}) = \{\bar{p}^{-1}\bar{x}\sigma(\bar{p})\bar{x}^{-1}; \bar{p} \in \bar{\mathcal{P}}\}.$$

Therefore, $\mathcal{P}\bar{x}\sigma(\mathcal{P}) = \mathcal{P} \cdot_{\sigma} I\bar{x}I$. Similarly, $\mathcal{P}\bar{x}\sigma(\mathcal{P}) = \mathcal{P} \cdot_{\sigma} I\bar{x}I$. Thus $\mathcal{P} \cdot_{\sigma} I\bar{x}I = \mathcal{P} \cdot_{\sigma} I\bar{x}I$ and $G(L) \cdot_{\sigma} I\bar{x}I = G(L) \cdot_{\sigma} I\bar{x}I$. \square

Now we describe $G(L) \cdot_{\sigma} I\bar{x}I$ for minimal length element \bar{x} .

Theorem 3.3. *If $\bar{x} \in \bar{W}$ is a minimal length element in its δ -conjugacy class, then $G(L) \cdot_{\sigma} I\bar{x}I$ is the single σ -conjugacy class $G(L) \cdot_{\sigma} \bar{x}$.*

Proof. By Theorem 2.3, $\bar{x} \approx_{\delta} ux$ for some $J \subseteq \tilde{S}$ with W_J finite, straight element $x \in \tilde{W}$ and $u \in W_J$ with $x \in {}^J\tilde{W}^{\delta(J)}$ and $x\delta(J) = J$. Then by Lemma 3.1, $G(L) \cdot_{\sigma} I\bar{x}I = G(L) \cdot_{\sigma} IuxI$. By Lemma 3.2, $G(L) \cdot_{\sigma} IuxI = G(L) \cdot_{\sigma} IxI$.

Let \mathcal{O} be the δ -conjugacy class of x . By Proposition 2.4, there exists $x_1 \in \mathcal{O}_{\min}$ and $y \in W^{J_0}$ such that $y^{-1}x_1y$ is a length 0 element in \tilde{W}_{J_0} and $\nu_{y^{-1}x_1y} = \nu_{\mathcal{O}}$ is dominant. Therefore x_1 is a fundamental (J_0, y, δ) -alcove in the sense of [11] and every element in Ix_1I is σ -conjugate under I to x_1 .

By §2.4 (2), $x \approx_{\delta} x_1$. Now

$$G(L) \cdot_{\sigma} I\bar{x}I = G(L) \cdot_{\sigma} IxI = G(L) \cdot_{\sigma} Ix_1I = G(L) \cdot_{\sigma} x_1$$

is a single σ -conjugacy class. Since $\bar{x} \in G(L) \cdot_{\sigma} I\bar{x}I$, $G(L) \cdot_{\sigma} I\bar{x}I = G(L) \cdot_{\sigma} \bar{x}$. \square

We also have the following disjointness result.

Proposition 3.4. *Let $\bar{w}, \bar{w}' \in \bar{W}$ be minimal length elements in their δ -conjugacy classes respectively. Then \bar{w} and \bar{w}' are in the same σ -conjugacy class of $G(L)$ if and only if $f(\bar{w}) = f(\bar{w}')$.*

Proof. By the proof of Theorem 3.3, there exists δ -straight elements $x, x' \in \tilde{W}$ with $f(\bar{w}) = f(x)$, $f(\bar{w}') = f(x')$, $\bar{w} \in G(L) \cdot_{\sigma} x$ and $\bar{w}' \in G(L) \cdot_{\sigma} x'$.

If $f(\bar{w}) = f(\bar{w}')$, then $f(x) = f(x')$ and by §2.4 (1), x and x' are in the same δ -conjugacy class of \tilde{W} . Therefore $G(L) \cdot_{\sigma} x = G(L) \cdot_{\sigma} x'$. Hence \bar{w} and \bar{w}' are in the same σ -conjugacy class of $G(L)$.

Now we prove the other direction. Notice that $\kappa(G(L) \cdot_{\sigma} \bar{w}) = \kappa(\bar{w})$ and $\kappa(G(L) \cdot_{\sigma} \bar{w}') = \kappa(\bar{w}')$. Thus if $\kappa(\bar{w}) \neq \kappa(\bar{w}')$, then $G(L) \cdot_{\sigma} \bar{w} \cap G(L) \cdot_{\sigma} \bar{w}' = \emptyset$.

Now assume that $\kappa(\bar{w}) = \kappa(\bar{w}')$ and $\bar{\nu}_{\bar{w}} \neq \bar{\nu}_{\bar{w}'}$. Then $\bar{\nu}_x \neq \bar{\nu}_{x'}$. If \bar{w} and \bar{w}' are in the same σ -conjugacy class of $G(L)$, then so are x and

\dot{x}' . We may assume that $\dot{x} = g\dot{x}'\sigma(g)^{-1}$ for some $g \in G(L)$, then for any $m \in \mathbb{N}$,

$$\dot{x}\sigma(\dot{x}) \cdots \sigma^{m-1}(\dot{x}) = g\dot{x}'\sigma(\dot{x}') \cdots \sigma^{m-1}(\dot{x}')\sigma^n(g)^{-1}.$$

There exists $n \in \mathbb{N}$ such that $\delta^n = 1$, $x\delta(x) \cdots \delta^{n-1}(x) = t^\mu$ and $x'\delta(x') \cdots \delta^{n-1}(x') = t^{\mu'}$ for some $\mu, \mu' \in P$. Then $\mu = n\nu_x$ and $\mu' = n\nu_{x'}$. Since $\bar{\nu}_x \neq \bar{\nu}_{x'}$, $\mu \notin W \cdot \mu'$. Assume that $g \in IzI$ for some $z \in \tilde{W}$. Then $I\epsilon^{k\mu}I \cap I\dot{z}I\epsilon^{k\mu'}I\dot{z}^{-1}I \neq \emptyset$ for all $k \in \mathbb{N}$.

Notice that $I\dot{z}I\epsilon^{k\mu'}I\dot{z}^{-1}I \subseteq \cup_{\tilde{y}, \tilde{y}' \leq z} I\tilde{y}\epsilon^{k\mu'}(\tilde{y}')^{-1}I$. So $t^{k\mu} = \tilde{y}t^{k\mu'}(\tilde{y}')^{-1}$ for some $\tilde{y}, \tilde{y}' \leq z$. Assume that $\tilde{y} = yt^\chi$ and $\tilde{y}' = y't^{\chi'}$ with $\chi, \chi' \in X_*(T)$ and $y, y' \in W$. Then $\tilde{y}t^{k\mu'}(\tilde{y}')^{-1} = t^{y(k\mu' + \chi - \chi')}y(y')^{-1}$. Hence $y = y'$ and $k\mu' + \chi - \chi' = ky^{-1}\mu$. By definition, $\ell(t^\chi) \leq \ell(\tilde{y}) + \ell(y) \leq \ell(z) + \ell(w_0)$. Similarly, $\ell(t^{\chi'}) \leq \ell(z) + \ell(w_0)$. Since $\mu \notin W \cdot \mu'$, then $\ell(t^{\mu' - y^{-1}\mu}) \geq 1$. Now

$$k \leq \ell(t^{k(\mu' - y^{-1}\mu)}) = \ell(t^{\chi' - \chi}) \leq \ell(t^{\chi'}) + \ell(t^\chi) \leq 2\ell(z) + 2\ell(w_0).$$

That is a contradiction. \square

Now we obtain a new proof of Kottwitz's classification of σ -conjugacy classes on $G(L)$.

Theorem 3.5. *For any straight δ -conjugacy class \mathcal{O} of \tilde{W} , we fix a minimal length representative $\tilde{w}_\mathcal{O}$. Then*

$$G(L) = \sqcup_{\mathcal{O}} G(L) \cdot_\sigma \dot{\tilde{w}}_\mathcal{O},$$

here \mathcal{O} runs over all the straight δ -conjugacy classes of \tilde{W} .

Proof. By Proposition 3.4, $\cup_{\mathcal{O}} G(L) \cdot_\sigma \dot{\tilde{w}}_\mathcal{O}$ is a disjoint union. Now we prove that for any $\tilde{w} \in \tilde{W}$, $I\dot{\tilde{w}}I \subseteq \sqcup_{\mathcal{O}} G(L) \cdot_\sigma \dot{\tilde{w}}_\mathcal{O}$. We argue by induction.

If \tilde{w} is a minimal length element in its δ -conjugacy class, then the statement follows from the proof of Theorem 3.3.

If \tilde{w} is not a minimal length element in its δ -conjugacy class, then by Theorem 2.1, there exists $i \in \tilde{S}$, $\tilde{w}', \tilde{w}'' \in \tilde{W}$ with $\tilde{w}' \approx_\delta \tilde{w}$, $\ell(\tilde{w}'') = \ell(\tilde{w}') - 2$ and $\tilde{w}' \xrightarrow{s_i} \tilde{w}''$. By Lemma 3.1,

$$I\dot{\tilde{w}}I \subseteq G(L) \cdot_\sigma I\dot{\tilde{w}}'I \subseteq G(L) \cdot_\sigma I\dot{s}_i\tilde{w}''I \cup G(L) \cdot_\sigma I\dot{\tilde{w}}''I.$$

Notice that $\ell(s_i\tilde{w}'), \ell(\tilde{w}'') < \ell(\tilde{w}') = \ell(\tilde{w})$. Now the statement follows from induction hypothesis on $s_i\tilde{w}'$ and \tilde{w}'' . \square

3.2. As a consequence of Theorem 3.5, any σ -conjugacy class of $G(L)$ contains a representative in \tilde{W} . For split groups, this was first proved in [10, Section 7] in a different way.

Moreover, the map $f : \tilde{W} \rightarrow B(\tilde{W}, \delta)$ is in fact the restriction of a map defined on $G(L)$, $b \mapsto (\bar{\nu}_b, \kappa(b))$. We call $\bar{\nu}_b$ the (dominant) Newton vector of b .

4. AFFINE DELIGNE-LUSZTIG VARIETIES

4.1. We first recall the definition and some properties on the (finite) Deligne-Lusztig varieties.

Let H be a connected reductive algebraic group over \mathbb{F}_q and F be the Frobenius morphism on H . We fix an F -stable maximal torus T and a F -stable Borel subgroup $B \supset T$. Let W be the finite Weyl group and \mathbb{S} be the set of simple reflections determined by (B, T) . The morphism F on G induces bijections on W and \mathbb{S} , which we still denote by F .

Following [6, Definition 1.4], the Deligne-Lusztig varieties associated to $w \in W$ is defined by

$$X_w = X_w^H = \{g \in G/B; g^{-1}F(g) \in BwB\}.$$

The finite group H^F acts on X_w in a natural way. We denote by $H^F \backslash X_w$ the space of orbits.

It is known that X_w is smooth of dimension $\ell(w)$. Moreover, we have the following result on Deligne-Lusztig variety associated to minimal length elements, which is essentially contained in [18].

Theorem 4.1. *Let $w \in W$ be a minimal length element in its F -conjugacy class. Then $H^F \backslash X_w$ is quasi-isomorphic to the orbit space of an affine space $\mathbf{k}^{\ell(w)}$ by an action of a finite torus.*

The case where w is elliptic is in [18, 4.3 (a)]. The general case is not stated explicitly in [18]. However, it can be trivially reduced to the elliptic case.

Let J be the minimal F -stable subset of \mathbb{S} such that $w \in W_J$ and M_J be the standard Levi subgroup of H of type J . Then w is an elliptic element in W_J and the statement follows from the elliptic case by using the fact that $H^F \backslash X_w^H \cong M_J^F \backslash X_w^{M_J}$ (see, e.g., [4, section 2]).

4.2. We recall the definition of affine Deligne-Lusztig varieties.

Let LG be the loop group associated to G . This is the ind-group scheme over \mathbf{k} which represents the functor $R \mapsto G(R((\epsilon)))$ on the category of k -algebras. Let $\mathcal{F}l = LG/I$ be the *fppf* quotient, which is represented by an ind-scheme, ind-projective over \mathbf{k} .

Following [33], we define affine Deligne-Lusztig varieties as follows.

For any $b \in G(L)$ and $\tilde{w} \in \tilde{W}$, set

$$X_{\tilde{w}}(b) = \{gI \in G(L)/I; g^{-1}b\sigma(g) \in I\tilde{w}I\} \subseteq \mathcal{F}l.$$

For $b \in G(L)$, define

$$\mathbb{J}_b = \{g \in G(L); g^{-1}b\sigma(g) = b\}.$$

Then \mathbb{J}_b acts on $X_{\tilde{w}}(b)$ on the left for any $\tilde{w} \in \tilde{W}$.

Now we recall the “reduction” à la Deligne and Lusztig (see [8]).

Proposition 4.2. *Let $x \in \tilde{W}$, and let $s \in \tilde{\mathbb{S}}$ be a simple affine reflection.*

- (1) If $\ell(sx\delta(s)) = \ell(x)$, then there exists a universal homeomorphism $X_x(b) \rightarrow X_{sx\delta(s)}(b)$.
- (2) If $\ell(sx\delta(s)) = \ell(x) - 2$, then $X_x(b)$ can be written as a disjoint union $X_x(b) = X_1 \sqcup X_2$ where X_1 is closed and X_2 is open, and such that there exist morphisms $X_1 \rightarrow X_{sx\delta(s)}(b)$ and $X_2 \rightarrow X_{sx}(b)$ which are compositions of a Zariski-locally trivial fiber bundle with one-dimensional fibers and a universal homeomorphism.

It is easy to see that

Lemma 4.3. *Let $\tilde{x} \in \tilde{W}$ and $\tau \in \Omega$. Then $X_{\tilde{x}}(b)$ is isomorphic to $X_{\tau\tilde{x}\delta(\tau)^{-1}}(b)$.*

As a consequence of Lemma 4.2 and Lemma 4.3, we have that

Corollary 4.4. *Let $\tilde{w}, \tilde{w}' \in \tilde{W}$ with $\tilde{w} \approx_\delta \tilde{w}'$. Then $X_{\tilde{w}}(b)$ and $X_{\tilde{w}'}(b)$ are universally homeomorphic.*

4.3. In the rest of this section, we study the space of orbits $\mathbb{J}_b \backslash X_{\tilde{w}}(b)$ for minimal length elements \tilde{w} . In order to do this, we first introduce the notion of fundamental elements.

Following [10, Definition 13.1.1], for $x \in \tilde{W}$, we say that x is σ -fundamental if every element of $I\dot{x}I$ is σ -conjugate under I to \dot{x} . Now we prove that

Proposition 4.5. *Every δ -straight element is σ -fundamental.*

Proof. Let \mathcal{O} be a straight δ -conjugacy class. Then by the proof of Theorem 3.3, there exists $x \in \mathcal{O}_{\min}$ such that x is σ -fundamental. Let $x' \in \mathcal{O}_{\min}$ be a δ -straight element, then by §2.4 (2), $x \approx_\delta x'$. Now the Proposition following from the following Lemma.

Lemma 4.6. *Let $x \in \tilde{W}$ and $s \in \tilde{S}$ with $\ell(sx\delta(s)) = \ell(x)$. If x is σ -fundamental, then so is $sx\delta(s)$.*

Proof. We prove the case that $sx < x$. The other cases can be proved in the same way.

Since x is σ -fundamental, the map $I \rightarrow I\dot{x}I$, $g \mapsto g\dot{x}\sigma(g)^{-1}$, is surjective. Notice that $I\dot{x}I \cong I\dot{s}I \times_I I\dot{s}\dot{x}I$. Thus the map

$$I \times I \rightarrow I\dot{s}I \times I\dot{s}\dot{x}I, \quad (g, g') \mapsto (g\dot{s}(g')^{-1}, g'\dot{s}\dot{x}\sigma(g)^{-1})$$

is also surjective.

The map $I\dot{s}I \times I\dot{s}\dot{x}I \rightarrow I\dot{s}\dot{x}I \times I\sigma(\dot{s})I$, $(h, h') \mapsto (h', \sigma(h))$ is surjective. Hence the map

$$I \times I \rightarrow I\dot{s}\dot{x}I \times I\sigma(\dot{s})I, \quad (g, g') \mapsto (g'\dot{s}\dot{x}\sigma(g)^{-1}, \sigma(g)\sigma(\dot{s})\sigma(g')^{-1})$$

is also surjective.

The quotient of $I\dot{x}I \times I\sigma(\dot{s})I$ by the action of the first I is $I\dot{x}I \times_I I\sigma(\dot{s})I \cong I\dot{x}\sigma(\dot{s})I$. Hence the map

$$I \mapsto I\dot{x}\sigma(\dot{s})I, \quad g' \mapsto g'\dot{x}\sigma(\dot{s})\sigma(g')^{-1}$$

is surjective. \square

Now we prove the main result of this section, which generalizes Theorem 4.1.

Theorem 4.7. *Let $\tilde{w} \in \tilde{W}$ be a minimal length element in its δ -conjugacy class. Let $b \in G(L)$ with $f(\tilde{w}) = f(b)$. Then $\dim X_{\tilde{w}}(b) = \ell(\tilde{w}) - \langle \bar{\nu}_b, 2\rho \rangle$ and $\mathbb{J}_b \backslash X_{\tilde{w}}(b)$ is in bijection with the orbit space of the affine space $\mathbf{k}^{\ell(\tilde{w}) - \langle \bar{\nu}_b, 2\rho \rangle}$ under an action of a finite torus.*

Proof. By Theorem 2.3, $\tilde{w} \approx_{\delta} ux$ for some $J \subseteq \tilde{\mathbb{S}}$ with W_J finite, straight element $x \in \tilde{W}$ and $u \in W_J$ with $x \in {}^J\tilde{W}^{\delta(J)}$ and $x\delta(J) = J$. By remark of Theorem 2.3, $f(\tilde{w}) = f(x) = f(b)$. Then \dot{x} and b are σ -conjugate and $\ell(x) = \langle \bar{\nu}_b, 2\rho \rangle$. Then $X_{\tilde{w}}(\dot{x})$ and $X_{\tilde{w}}(b)$ are isomorphic and we have a natural bijection $\mathbb{J}_{\dot{x}} \backslash X_{\tilde{w}}(\dot{x}) \cong \mathbb{J}_b \backslash X_{\tilde{w}}(b)$. By Corollary 4.4, $X_{\tilde{w}}(\dot{x})$ and $X_{ux}(\dot{x})$ are universally homeomorphic.

Let \mathcal{P} be the standard parahoric subgroup corresponding to J and \mathcal{U} be its prounipotent radical. Let $g \in G(L)$ with $g^{-1}\dot{x}\sigma(g) \in I\dot{x}I$. By the proof of Lemma 3.2, there exists $p \in \mathcal{P}$ such that $(gp)^{-1}\dot{x}\sigma(gp) \in I\dot{x}I$. By Proposition 4.5, there exists $p' \in I$ such that $(gpp')^{-1}\dot{x}\sigma(gpp') = \dot{x}$. Hence $g \in \mathbb{J}_{\dot{x}}\mathcal{P}$.

Notice that $\mathbb{J}_{\dot{x}}\mathcal{P}$ is in bijection with the quotient space $\mathbb{J}_{\dot{x}} \times_{\mathbb{J}_{\dot{x}} \cap \mathcal{P}} \mathcal{P}$. Then we have a natural bijection

$$X_{ux}(\dot{x}) \cong \mathbb{J}_{\dot{x}} \times_{\mathbb{J}_{\dot{x}} \cap \mathcal{P}} X_{ux}^{\mathcal{P}}(\dot{x}),$$

where $X_{ux}^{\mathcal{P}}(\dot{x}) = \{g \in \mathcal{P}/I; g^{-1}\dot{x}\sigma(g) \in I\dot{x}I\}$. In particular, $\mathbb{J}_b \backslash X_{\tilde{w}}(b) \cong (\mathbb{J}_{\dot{x}} \cap \mathcal{P}) \backslash X_{ux}^{\mathcal{P}}(\dot{x})$.

Moreover, the projection $\mathcal{F}l \rightarrow LG/\mathcal{P}$ sends $X_{ux}(\dot{x})$ onto $\mathbb{J}_{\dot{x}}/(\mathbb{J}_{\dot{x}} \cap \mathcal{P})$ and each fiber is isomorphic to $X_{ux}^{\mathcal{P}}(\dot{x})$. Hence $\dim X_{\tilde{w}}(b) = \dim X_{ux}(\dot{x}) = \dim X_{ux}^{\mathcal{P}}(\dot{x})$.

Notice that $\bar{I} = I/\mathcal{U}$ is a Borel subgroup of $\bar{\mathcal{P}}$. Therefore $X_{ux}^{\mathcal{P}}(\dot{x})$ is isomorphic to (finite) Deligne-Lusztig variety

$$X' = \{\bar{p} \in \bar{\mathcal{P}}/\bar{I}; \bar{p}^{-1}\sigma_{\dot{x}}(\bar{p}) \in \bar{I}\dot{u}\bar{I}\}/\bar{I}$$

of $\bar{\mathcal{P}}$. The action of $\mathbb{J}_{\dot{x}} \cap \mathcal{P}$ on $X_{ux}^{\mathcal{P}}(\dot{x})$ factors through an action of $(\mathbb{J}_{\dot{x}} \cap \mathcal{P})\mathcal{U}/\mathcal{U} \cong \bar{\mathcal{P}}^{\sigma_{\dot{x}}}$, here $\sigma_{\dot{x}}$ is the Frobenius morphism on $\bar{\mathcal{P}}$ defined by $\bar{p} \mapsto \dot{x}\sigma(\bar{p})\dot{x}^{-1}$. Hence we have a natural bijection $(\mathbb{J}_{\dot{x}} \cap \mathcal{P}) \backslash X_{ux}^{\mathcal{P}}(\dot{x}) \cong \bar{\mathcal{P}}^{\sigma_{\dot{x}}} \backslash X'$.

The map $v \mapsto vx$ sends a $\sigma_{\dot{x}}$ -conjugacy class in W_J into a δ -conjugacy class in \tilde{W} . Since ux is a minimal length element in its δ -conjugacy class, u is a minimal length element in its $\sigma_{\dot{x}}$ -conjugacy class in W_J . Moreover, $\ell(u) = \ell(ux) - \ell(x) = \ell(\tilde{w}) - \langle \bar{\nu}_b, 2\rho \rangle$. Now the Theorem follows from Theorem 4.1. \square

By the same argument, we have the following result.

Proposition 4.8. *Let $J \subseteq \tilde{\mathbb{S}}$ with W_J finite, $x \in {}^J\tilde{W}^{\delta(J)}$ with $x\delta(J) = J$ and $u \in W_J$. Then for any $b \in G(L)$,*

$$\dim X_{ux}(b) = \dim X_x(b) + \ell(u).$$

5. HOMOLOGY OF AFFINE DELIGNE-LUSZTIG VARIETIES

5.1. Notice that $X_{\tilde{w}}(b) = \lim_{\rightarrow} X_i$ for some closed subschemes $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{\tilde{w}}(b)$ of finite type. Let l be a prime with l not equal to the characteristic of \mathbf{k} . Then $H_c^j(X_i, \bar{\mathbb{Q}}_l)$ is defined for all j . Set

$$H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l) = \lim_{\rightarrow} H_c^j(X_i, \bar{\mathbb{Q}}_l)^*.$$

Then $H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l)$ is a smooth representation of \mathbb{J}_b . Hence it is a semisimple module for any open compact subgroup of \mathbb{J}_b .

The following result can be proved along the line of [6, Theorem 1.6].

Lemma 5.1. *Let $b \in G(L)$ and K be an open compact subgroup of \mathbb{J}_b . Let $\tilde{w} \in \tilde{W}$, and let $i \in \tilde{\mathbb{S}}$. Then*

(1) *If $\ell(s_i \tilde{w} s_{\delta(i)}) = \ell(\tilde{w})$, then for any $j \in \mathbb{Z}$,*

$$H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l) \cong H_j^{BM}(X_{s_i \tilde{w} s_{\delta(i)}}(b), \bar{\mathbb{Q}}_l)$$

as \mathbb{J}_b -modules.

(2) *If $\ell(s_i \tilde{w} s_{\delta(i)}) = \ell(\tilde{w}) - 2$, then for any simple K -module M that is a direct summand of $\oplus_j H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l)$, M is also a direct summand of $\oplus_j H_j^{BM}(X_{s_i \tilde{w} s_{\delta(i)}}(b), \bar{\mathbb{Q}}_l) \oplus \oplus_j H_j^{BM}(X_{s_i \tilde{w}}(b), \bar{\mathbb{Q}}_l)$.*

Now we can show the following “finiteness” result.

Theorem 5.2. *Let $b \in G(L)$ and K be an open compact subgroup of \mathbb{J}_b . Let M be a simple K -module. If M is a direct summand of $\oplus_{\tilde{w} \in \tilde{W}} \oplus_j H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l)$, then M is a direct summand of $H_j^{BM}(X_{\tilde{x}}(b), \bar{\mathbb{Q}}_l)$, where $j \in \mathbb{Z}$ and \tilde{x} is a minimal length element in its δ -conjugacy class and $f(\tilde{x}) = f(b)$.*

Remark. Given b , there are only finitely many \tilde{x} satisfying the conditions above.

Proof. Let \tilde{w} be a minimal length element in \tilde{W} such that M is a direct summand of $\oplus_j H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l)$. By Theorem 2.1 and Lemma 5.1, \tilde{w} is a minimal length element in its δ -conjugacy class. By Theorem 3.3, $G(L) \cdot_{\sigma} I \tilde{w} I$ is a single σ -conjugacy class. Hence $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $b \in G(L) \cdot_{\sigma} I \tilde{w} I$, i.e. $f(b) = f(\tilde{w})$. \square

In the rest of this section, we discuss the special case where b is a superstraight element. We first describe its centralizer in the affine Weyl group and in the loop group.

Proposition 5.3. *Let $J \subseteq S$ and x be a δ -superbasic element in \tilde{W}_J with $\nu_x \in P_{\mathbb{Q},+}$ and $J = \{i \in S; \langle \nu_x, \alpha_i \rangle = 0\}$.*

(1) *Let $Z_{\tilde{W},\delta}(x) = \{w \in \tilde{W}; w^{-1}x\delta(w) = x\}$ be the δ -centralizer of x in \tilde{W} . Then $Z_{\tilde{W},\delta}(x)$ consists of length 0 elements y in \tilde{W}_J with $y^{-1}x\delta(y) = x$.*

(2) *$M_J(L) \cap I \cap \mathbb{J}_{\dot{x}}$ is a normal subgroup of $\mathbb{J}_{\dot{x}}$ and*

$$\mathbb{J}_{\dot{x}} / (M_J(L) \cap I \cap \mathbb{J}_{\dot{x}}) \cong Z_{\tilde{W},\delta}(x).$$

Proof. Let $n \in \mathbb{N}$ with $\delta^n = id$ and $x\delta(x) \cdots \delta^{n-1}(x) = t^{n\nu_x}$. Hence $\dot{x}\sigma(\dot{x}) \cdots \sigma^{n-1}(\dot{x}) \in \epsilon^{n\nu_x}T(L)_1$. Thus after replacing \dot{x} by $h^{-1}\dot{x}\sigma(h)$ for a suitable $h \in T(L)_1$, we may assume that $\dot{x}\sigma(\dot{x}) \cdots \sigma^{n-1}(\dot{x}) = \epsilon^{n\nu_x}$.

(1) Let $w \in \tilde{W}$ with $w^{-1}x\delta(w) = x$. Then

$$\begin{aligned} w^{-1}t^{n\nu_x}w &= w^{-1}x\delta(x) \cdots \delta^{n-1}(x)\delta^n(w) \\ &= (w^{-1}x\delta(w))\delta(w^{-1}x\delta(w)) \cdots \delta^{n-1}(w^{-1}x\delta(w)) \\ &= x\delta(x) \cdots \delta^{n-1}(x) = t^{n\nu_x}. \end{aligned}$$

Thus $w\nu_x = \nu_x$ and $w \in \tilde{W}_J$.

Let \tilde{S}' be the set of simple reflection in \tilde{W}_J . Let ℓ_J be the length function and $<_J$ be the Bruhat order on \tilde{W}_J . If $\ell_J(w) > 0$, then there exists $i \in \tilde{S}'$ such that $s_i w <_J w$. Since the map $\tilde{W}_J \rightarrow \tilde{W}_J$, $y \mapsto x\delta(y)x^{-1}$ preserve the Bruhat order, we have that

$$xs_{\delta(i)}x^{-1}w = xs_{\delta(i)}\delta(w)x^{-1} <_J x\delta(w)x^{-1} = w.$$

Similarly, $(\text{Ad}(x) \circ \delta)^m(s_i)w <_J w$ for all $m \in \mathbb{Z}$. However, x is δ -superbasic for \tilde{W}_J and thus \tilde{S}' is a single orbit of $\text{Ad}(x) \circ \delta$. Hence $s_j w < w$ for all $j \in \tilde{S}'$. This is impossible. Thus $\ell_J(w) = 0$.

Part (1) is proved.

(2) Let $g \in \mathbb{J}_{\dot{x}}$. Then $g^{-1}\dot{x}\sigma(g) = \dot{x}$. We prove that

(a) $g \in M_J(L)$.

Similar to part (1), we have that $g^{-1}\epsilon^{n\nu_x}\sigma^n(g) = \epsilon^{n\nu_x}$ and $\epsilon^{-n\nu_x}g\epsilon^{n\nu_x} = \sigma^n(g)$. Let $U(L)$ be the subgroup generated by U_a for $a \in \Phi^+ - \Phi_J^+$. We may assume that $g = um\dot{x}m'u'$ for $u, u' \in U(L)$, $m, m' \in M_J(L)$ and $w \in {}^JW^J$. Since ν_x is neutral for $M_J(L)$, we have that

$$(\epsilon^{-n\nu_x}u\epsilon^{n\nu_x})m(\epsilon^{-n\nu_x}\dot{x}\epsilon^{n\nu_x})m'(\epsilon^{-n\nu_x}u'\epsilon^{n\nu_x}) = \sigma^n(u)\sigma^n(m)\sigma^n(\dot{x})\sigma^n(m')\sigma^n(u').$$

Therefore $m(\epsilon^{-n\nu_x}\dot{x}\epsilon^{n\nu_x})m' = \sigma^n(m\dot{x}m')$.

Let $M'(L)$ be the derived group of $M_J(L)$ and Z be the center of $M_J(L)$. Then $Z \subseteq T(L)$ and $M_J(L) = M'(L)Z$. We may write m as m_1z and m' as m'_1z' for $m, m' \in M'$ and $z, z' \in Z$. Then we have that

$$z(\epsilon^{-n\nu_x}\dot{x}\epsilon^{n\nu_x})z' = z\epsilon^{n(w\nu_x - \nu_x)}(\dot{x}z'z'^{-1})\dot{x} = \sigma^n(z\dot{x}z').$$

Notice that $z\dot{w}z'\dot{w}^{-1} \in T(L)$. Hence

$$\epsilon^{n(w\nu_x - \nu_x)} = (z\dot{w}z'\dot{w}^{-1})^{-1}\sigma^n(z\dot{w}z'\dot{w}^{-1})(\sigma^n(\dot{w})\dot{w}^{-1}).$$

We have that $(z\dot{w}z'\dot{w}^{-1})^{-1}\sigma^n(z\dot{w}z'\dot{w}^{-1}), \sigma^n(\dot{w})\dot{w}^{-1} \in T(L)_1$. Thus $\epsilon^{n(w\nu_x - \nu_x)} \in T(L)_1$ and $w\nu_x = \nu_x$. Therefore $\langle \nu_x, w^{-1}\alpha \rangle = \langle w\nu_x, \alpha \rangle = \langle \nu_x, \alpha \rangle = 0$ for all $\alpha \in \Phi_J$. Therefore $w^{-1}\alpha \in \Phi_J$ for all $\alpha \in \Phi_J$ and $w = 1$. Hence $g = um$ for some $u \in U(L)$ and $m \in M_J(L)$ and $\epsilon^{-n\nu_x}u\epsilon^{n\nu_x} = \sigma^n(u)$. Notice that $\langle \nu_x, \alpha \rangle > 0$ for all $\alpha \in \Phi^+ - \Phi_J^+$, we must have that $u = 1$.

Thus $g \in M_J(L)$ and (a) is proved.

Now set $I' = I \cap M_J(L)$. Then we may assume that $g \in I'\dot{y}I'$ for some $y \in \tilde{W}_J$. Since x is a length 0 element in \tilde{W}_J , $\dot{x}I' = I'\dot{x}$. Thus $g = \dot{x}\sigma(g)\dot{x}^{-1} \in \dot{x}I'\sigma(\dot{y})I'\dot{x}^{-1} = I'\dot{x}\sigma(\dot{y})\dot{x}^{-1}I'$ and $I'\dot{y}I' \cap I'\dot{x}\sigma(\dot{y})\dot{x}^{-1}I' \neq \emptyset$. Therefore $y = x\delta(y)x^{-1}$ and $y \in Z_{\tilde{W},\delta}(x)$ is a length 0 element in \tilde{W}_J . Thus $g \in I'\dot{y}$ and

$$g^{-1}(\mathbb{J}_{\dot{x}} \cap I')g \subseteq \mathbb{J}_{\dot{x}} \cap g^{-1}I'g = \mathbb{J}_{\dot{x}} \cap \dot{y}^{-1}I'\dot{y} = \mathbb{J}_{\dot{x}} \cap I'.$$

Hence $\mathbb{J}_{\dot{x}} \cap I'$ is a normal subgroup of $\mathbb{J}_{\dot{x}}$. Moreover, we have an injective group homomorphism

$$\pi : \mathbb{J}_{\dot{x}}/(\mathbb{J}_{\dot{x}} \cap I') \rightarrow Z_{\tilde{W},\delta}(x).$$

On the other hand, let $y \in Z_{\tilde{W},\delta}(x)$. Then $y^{-1}x\delta(y) = x$ and $\dot{y}^{-1}\dot{x}\sigma(\dot{y}) \in \dot{x}T(L)_1$. Replacing \dot{y} by $\dot{y}h$ for a suitable element $h \in T(L)_1$, we have that $\dot{y}^{-1}\dot{x}\sigma(\dot{y}) = \dot{x}$. Hence the map π is surjective and $\mathbb{J}_{\dot{x}}/(\mathbb{J}_{\dot{x}} \cap I') \cong Z_{\tilde{W},\delta}(x)$. \square

Theorem 5.4. *Let $x \in \tilde{W}$ be a δ -superstraight element. Then for any $\tilde{w} \in \tilde{W}$ with $X_{\tilde{w}}(\dot{x}) \neq \emptyset$ and $j \in \mathbb{Z}$, the action of $\mathbb{J}_{\dot{x}}$ on $H_j^{BM}(X_{\tilde{w}}(b), \bar{\mathbb{Q}}_l)$ factors through an action of the δ -centralizer $Z_{\tilde{W},\delta}(x)$ of x .*

Proof. By Proposition 2.5, there exists a superbasic element $x_1 \in \tilde{W}_{J_0}$ and $y \in W^{J_0}$ such that $\nu_{x_1} = \nu_0$ and $yx_1\delta(y)^{-1} \in \mathcal{O}_{\min}$.

After σ -conjugating by a suitable element in $G(L)$, we may assume that $\dot{x} = \dot{y}\dot{x}_1\sigma(\dot{y})^{-1}$. Then $\mathbb{J}_{\dot{x}} = \dot{y}\mathbb{J}_{\dot{x}_1}\dot{y}^{-1}$.

By Proposition 5.3, $M_{J_0}(L) \cap I \cap \mathbb{J}_{\dot{x}_1}$ is a normal subgroup of $\mathbb{J}_{\dot{x}_1}$ and $\mathbb{J}_{\dot{x}_1}/(M_{J_0}(L) \cap I \cap \mathbb{J}_{\dot{x}_1}) \cong Z_{\tilde{W},\delta}(x_1)$. Set $K = \dot{y}(M_{J_0}(L) \cap I)\dot{y}^{-1} \cap \mathbb{J}_{\dot{x}}$. Then K is a normal subgroup of $\mathbb{J}_{\dot{x}} = \dot{y}\mathbb{J}_{\dot{x}_1}\dot{y}^{-1}$ and $\mathbb{J}_{\dot{x}}/K \cong yZ_{\tilde{W},\delta}(x_1)y^{-1} = Z_{\tilde{W},\delta}(x)$.

By the proof of Theorem 4.7, $X_x(\dot{x}) = \mathbb{J}_{\dot{x}}I/I \cong \mathbb{J}_{\dot{x}}/(\mathbb{J}_{\dot{x}} \cap I)$. Since $y \in W^{J_0}$, $\dot{y}(M_{J_0}(L) \cap I)\dot{y}^{-1} \subseteq I$. Thus K acts trivially on $X_x(\dot{x})$ and also trivially on $H_j^{BM}(X_x(\dot{x}), \bar{\mathbb{Q}}_l)$ for any $j \in \mathbb{Z}$. By Theorem 5.2 and Lemma 5.1 (1), any simple K -module that appears as a direct summand of $H_j^{BM}(X_{\tilde{w}}(\dot{x}), \bar{\mathbb{Q}}_l)$ is trivial. Hence K acts trivially on $H_j^{BM}(X_{\tilde{w}}(\dot{x}), \bar{\mathbb{Q}}_l)$ and the action of $\mathbb{J}_{\dot{x}}$ factor through $Z_{\tilde{w},\delta}(x)$. \square

5.2. Now we discuss some special cases.

Let $\lambda \in P^\delta$ be a regular element and \mathcal{O} be the δ -conjugacy class that contains t^λ . Then $\nu_{\mathcal{O}} = \lambda$ is regular. By §2.7, \mathcal{O} is superstraight. By §2.4, t^λ is δ -superstraight since $\ell(t^\lambda) = \langle \lambda, 2\rho \rangle$. Hence t^λ is δ -superstraight and by Proposition 5.3, $Z_{\tilde{W}, \delta}(t^\lambda) = \{t^\mu; \mu = \delta(\mu) \in P\} \cong P^\delta$. Therefore by Theorem 5.4, for any $\tilde{w} \in \tilde{W}$ and $j \in \mathbb{Z}$, the action of $\mathbb{J}_{\epsilon^\lambda}$ on $H_j^{BM}(X_{\tilde{w}}(\epsilon^\lambda), \bar{\mathbb{Q}}_l)$ factors through an action of P^δ . The special case for split SL_2 and SL_3 was first obtained by Zbarsky [39] via direct calculation.

Let τ be a δ -superbasic element. Then τ is δ -superstraight and by Proposition 5.3, $Z_{\tilde{W}, \delta}(\tau) = \Omega^\delta$. Therefore by Theorem 5.4, for any $\tilde{w} \in \tilde{W}$ and $j \in \mathbb{Z}$, the action of \mathbb{J}_τ on $H_j^{BM}(X_{\tilde{w}}(\tau), \bar{\mathbb{Q}}_l)$ factors through an action of Ω^δ .

6. DIMENSION “=” DEGREE

In this section, we give a formula which relates the dimension of affine Deligne-Lusztig varieties with the degree of the class polynomials.

This is a key part of this paper. On one hand, it provides both theoretic and practical way to determine the dimension of affine Deligne-Lusztig varieties. On the other hand, it showed that the dimension and nonemptiness pattern of affine Deligne-Lusztig varieties $X_{\tilde{w}}(b)$ only depends on the data $(\tilde{W}, \delta, \tilde{w}, f(b))$ and thus independent of the choice of G . In other words, it suffices to consider the case where G is split over L .

Theorem 6.1. *Let $b \in G(L)$ and $\tilde{w} \in \tilde{W}$. Then*

$$\dim(X_{\tilde{w}}(b)) = \max_{\mathcal{O}} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathcal{O}) + \deg(f_{\tilde{w}, \mathcal{O}})) - \langle \bar{\nu}_b, 2\rho \rangle,$$

here \mathcal{O} runs over δ -conjugacy class of \tilde{W} with $f(\mathcal{O}) = f(b)$ and $\ell(\mathcal{O})$ is the length of any minimal length element in \mathcal{O} .

Proof. We argue by induction on $\ell(\tilde{w})$.

Let \mathcal{O}' be the δ -conjugacy class that contains \tilde{w} .

If $\tilde{w} \in \mathcal{O}'_{\min}$, then

$$\frac{1}{2}(\ell(\tilde{w}) + \ell(\mathcal{O}) + \deg(f_{\tilde{w}, \mathcal{O}})) = \begin{cases} \ell(\tilde{w}), & \text{if } \mathcal{O} = \mathcal{O}'; \\ -\infty, & \text{if } \mathcal{O} \neq \mathcal{O}'. \end{cases}$$

Thus the right hand side of (a) is $-\infty$ if $f(\tilde{w}) \neq f(b)$ and $\ell(\tilde{w}) - \langle \bar{\nu}_b, 2\rho \rangle$ if $f(\tilde{w}) = f(b)$. Now the statement follows from Theorem 4.7.

If $\tilde{w} \notin \mathcal{O}'_{\min}$, by Theorem 2.1 there exists $\tilde{w}_1 \approx_\delta \tilde{w}$ and $i \in \tilde{S}$ such that $\ell(s_i \tilde{w}_1 s_{\delta(i)}) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. Then by §2.3, $f_{\tilde{w}, \mathcal{O}} = (v - v^{-1})f_{s_i \tilde{w}_1, \mathcal{O}} + f_{s_i \tilde{w}_1 s_{\delta(i)}, \mathcal{O}}$. Hence $\deg(f_{\tilde{w}, \mathcal{O}}) = \max\{\deg(f_{s_i \tilde{w}_1, \mathcal{O}}) + 1, \deg(f_{s_i \tilde{w}_1 s_{\delta(i)}, \mathcal{O}})\}$ and $(\ell(\tilde{w}) + \deg(f_{\tilde{w}, \mathcal{O}})) = \max\{(\ell(s_i \tilde{w}_1) + \deg(f_{s_i \tilde{w}_1, \mathcal{O}})), (\ell(s_i \tilde{w}_1 s_{\delta(i)}) + \deg(f_{s_i \tilde{w}_1 s_{\delta(i)}, \mathcal{O}}))\}$.

$\deg(f_{s_i \tilde{w}_1 s_{\delta(i)}, \mathcal{O}}))\} + 1$. By Lemma 4.2, $\dim(X_{\tilde{w}}(b)) = \dim(X_{\tilde{w}_1}(b)) = \max\{\dim(X_{s_i \tilde{w}_1, \delta_F}(b)), \dim(X_{s_i \tilde{w}_1 s_{\delta(i)}}(b))\} + 1$. Now the statement follows from induction hypothesis. \square

Corollary 6.2. *Let $b \in G(L)$ and $\tilde{w} \in \tilde{W}$. Then $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $f_{\tilde{w}, \mathcal{O}} \neq 0$ for some δ -conjugacy class \mathcal{O} of \tilde{W} with $f(\mathcal{O}) = f(b)$.*

Corollary 6.3. *Let \mathcal{O} be a superstraight δ -conjugacy class in \tilde{W} and $x \in \mathcal{O}_{\min}$ is a δ -superstraight element. Then for any $\tilde{w} \in \tilde{W}$,*

$$\dim(X_{\tilde{w}}(\dot{x})) = \frac{1}{2}(\ell(\tilde{w}) + \deg(f_{\tilde{w}, \mathcal{O}})) - \langle \nu_{\mathcal{O}}, \rho \rangle.$$

In particular, $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $f_{\tilde{w}, \mathcal{O}} \neq 0$.

6.1. By Theorem 6.1 and its consequences, we obtain all the information on the emptiness/nonemptiness and dimension formula of affine Deligne-Lusztig varieties if the degree of the class polynomials are known. The latter problem requires a thorough understanding of trace formula of all finite dimensional representation of affine Hecke algebras and thus computing class polynomials is quite hard in general and not known yet.

However, at present, Theorem 6.1 is still quite useful in the study of affine Deligne-Lusztig varieties. It implies that the emptiness/nonemptiness and dimension formula of affine Deligne-Lusztig varieties only rely on the reduction method. Such observation will play a key role in the proof of emptiness/nonemptiness pattern of affine Deligne-Lusztig varieties for basic b , which will be discussed in [11].

6.2. One can show that the reduction method also works in the p -adic case and the p -adic variant of $X_{\tilde{w}}(b)$ (for $b \in \tilde{W}$) is nonempty if and only if $f_{\tilde{w}, \mathcal{O}} \neq 0$ for some δ -conjugacy class \mathcal{O} of \tilde{W} with $f(\mathcal{O}) = f(b)$.

There is no known good notion of dimension for the p -adic variant of affine Deligne-Lusztig varieties. However, one may hope that once it is established, then the dimension of $X_{\tilde{w}}(b)$ should agree in the p -adic and function field case, and thus the “dimension=degree” theorem remains valid for p -adic case.

7. MAZUR’S INEQUALITY AND ITS CONVERSE

7.1. In this section, we discuss some application of Theorem 6.1 to the hyperspecial case.

Let \mathcal{G} be the smooth affine group scheme associated to the special vertex of the Bruhat-Tits building of G and $L^+\mathcal{G}$ be the infinite-dimensional affine group scheme defined by $L^+\mathcal{G}(R) = \mathcal{G}(R[[\epsilon]])$. The fpqc quotient $Gr = LG/L^+\mathcal{G}$ is called the (twisted) affine Grassmannian. We have the Cartan decomposition (see [35]).

$$\begin{aligned} G(L) &= \sqcup_{\mu \in P_+} L^+\mathcal{G}(\mathbf{k})\epsilon^\mu L^+\mathcal{G}(\mathbf{k}), \\ Gr(k) &= \sqcup_{\mu \in P_+} L^+\mathcal{G}(\mathbf{k})\epsilon^\mu L^+\mathcal{G}(\mathbf{k})/L^+\mathcal{G}(\mathbf{k}). \end{aligned}$$

The affine Deligne-Lusztig variety associated with $b \in G(L)$ and $\mu \in P_+$ is given by

$$X_\mu(b) = \{g \in G(L); g^{-1}b\sigma(g) \in L^+\mathcal{G}(\mathbf{k})\epsilon^\mu L^+\mathcal{G}(\mathbf{k})\}/L^+\mathcal{G}(\mathbf{k}) \subseteq Gr(k).$$

7.2. Let $J \subseteq \mathbb{S}$ with $\delta(J) = J$. Let X_J be the quotient of P by the sublattice of Q generated by the simple coroots in J . The action of δ extends in a natural way to X_J . Let $Y_J = X_J/(1 - \delta)X_J$ be the coinvariants of this action and Y_J^+ be the image of P_+ in Y_J .

Following [7], we define a partial order \preceq_J on Y_J^+ as follows. For $\mu, \mu' \in Y_J$, we write $\mu \preceq_J \mu'$ if $\mu' - \mu$ is a nonnegative integral linear combination of the image in Y_J of the simple coroots in $\mathbb{S} - J$.

Now we have the following result.

Theorem 7.1. *Let $J \subseteq \mathbb{S}$ with $\delta(J) = J$. Let $\mu \in P_+$ and $b \in M_J(L)$ be a basic element such that $\kappa_{M_J}(b) \in Y_J^+$. Then $X_\mu(b) \neq \emptyset$ if and only if $\kappa_{M_J}(b) \preceq_J \mu$.*

Remark. The “only if” side was proved by Rapoport and Richartz in [34], which generalize Mazur’s theorem. The “if” side was conjectured by Kottwitz and Rapoport and proved for type A and C in [28]. It was then proved by Lucarelli [31] for classical split groups and then by Gashi [7] for unramified cases.

Proof. Since $L^+\mathcal{G}(\mathbf{k})t^\mu L^+\mathcal{G}(\mathbf{k}) = \sqcup_{\tilde{w} \in Wt^\mu W} I\tilde{w}I$, $X_\mu(b) \neq \emptyset$ if and only if $X_{\tilde{w}}(b) \neq \emptyset$ for some $\tilde{w} \in Wt^\mu W$. Let C be the set of δ -conjugacy classes \mathcal{O} in \tilde{W} with $f(\mathcal{O}) = f(b)$. Then C only depends on $\kappa_{M_J}(b)$. Thus by Theorem 6.1,

(a) $X_\mu(b) \neq \emptyset$ if and only if $f_{\tilde{w}, \mathcal{O}} \neq 0$ for some $\tilde{w} \in Wt^\mu W$ and $\mathcal{O} \in C$.

The latter one only depends on the combinatorial data $(\tilde{W}, \delta, \mu, J, \kappa_{M_J}(b))$ and is independent of the loop group G .

To the pair (\tilde{W}, δ) , we may associate a quasi-split unramified group H . By [7, Theorem 5.1], the statement holds for H . Thus by (a), we have that

(b) $\kappa_{M_J}(b) \preceq_J \mu$ if and only if $f_{\tilde{w}, \mathcal{O}} \neq 0$ for some $\tilde{w} \in Wt^\mu W$ and $\mathcal{O} \in C$.

The theorem then follows from (a) and (b). \square

8. SUPERBASIC ELEMENTS

For $b \in G(L)$, if $X_{\tilde{w}}(b)$ is nonempty, then usually it has infinitely many irreducible components. However, for superbasic elements, the affine Deligne-Lusztig variety is much nicer.

Proposition 8.1. *Assume that G is semisimple and $x \in \tilde{W}$ is a δ -superbasic element. Then for any $\tilde{w} \in \tilde{W}$, $X_{\tilde{w}}(x)$ has only finitely many irreducible components.*

Proof. We argue by induction on $\ell(\tilde{w})$.

Let \mathcal{O} be the δ -conjugacy class containing \tilde{w} .

If $\tilde{w} \in \mathcal{O}_{\min}$, then by Theorem 3.3 and Proposition 3.4, $X_{\tilde{w}}(\dot{x}) \neq \emptyset$ if and only if $f(\tilde{w}) = f(x)$. Since x is δ -superbasic, we have that \tilde{w} is δ -conjugate to x . By the proof of Theorem 5.4, $X_{\tilde{w}}(\dot{x}) \cong X_{\tilde{w}}(\dot{\tilde{w}})$ can be identified with a subset of Ω . Hence $X_{\tilde{w}}(\dot{x})$ is a finite set.

If $\tilde{w} \notin \mathcal{O}_{\min}$, then by Theorem 2.1 there exists $\tilde{w}_1 \approx_{\delta} \tilde{w}$ and $i \in \tilde{\mathbb{S}}$ such that $\ell(s_i \tilde{w}_1 s_{\delta(i)}) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. By induction hypothesis $X_{s_i \tilde{w}_1 s_{\delta(i)}}(\dot{x})$ and $X_{s_i \tilde{w}_1}(\dot{x})$ both have only finitely many irreducible components. Therefore by Proposition 4.2, $X_{\tilde{w}}(\dot{x})$ is universal homeomorphic to $X_{\tilde{w}_1}(\dot{x})$ and has only finitely many irreducible components. \square

Corollary 8.2. *Assume that G is semisimple and $x \in \tilde{W}$ is a δ -superbasic element. Then for any $\mu \in P_+$, $X_{\mu}(\dot{x})$ has only finitely many irreducible components.*

Remark. The split case was first proved by Viehmann [37] in a different way.

Proof. Let $\pi : \mathcal{F}l \rightarrow Gr$ be the projection. Then $\pi^{-1}X_{\mu}(\dot{x}) = \sqcup_{\tilde{w} \in W t^{\mu} W} X_{\tilde{w}}(\dot{x})$ is a finite union. Since each $X_{\tilde{w}}(\dot{x})$ has only finite many irreducible components, $\pi^{-1}X_{\mu}(\dot{x})$ has only finitely many irreducible components and so is $X_{\mu}(\dot{x})$. \square

8.1. Since for superbasic element b , $X_{\tilde{w}}(b)$ contains only finitely many irreducible components, the number of rational points is finite. Now we show that for split groups, there is a simple formula for this number in terms of class polynomials.

Proposition 8.3. *Assume that $G = PGL_n$ is split over $F = \mathbb{F}_q((\epsilon))$ and x is a superbasic element in \tilde{W} . Then for any $\tilde{w} \in \tilde{W}$,*

$$\sharp X_{\tilde{w}}(\dot{x})(\mathbb{F}_q) = nq^{\frac{\ell(\tilde{w})}{2}} f_{\tilde{w}, \mathcal{O}}|_{v=\sqrt{q}},$$

where \mathcal{O} is the conjugacy class of \tilde{W} that contains x .

Proof. We argue by induction on $\ell(\tilde{w})$.

Let \mathcal{O}' be the δ -conjugacy class that contains \tilde{w} .

If $\tilde{w} \in \mathcal{O}'_{\min}$, then $\mathcal{O}' = \mathcal{O}$ and \tilde{w} is conjugate to x . By the proof of Theorem 5.4, $X_{\tilde{w}}(\dot{x})(\mathbb{F}_q) \cong \Omega$. Hence $\sharp X_{\tilde{w}}(\dot{x}) = n$.

If $\tilde{w} \notin \mathcal{O}'_{\min}$, by Theorem 2.1 there exists $\tilde{w}_1 \approx \tilde{w}$ and $i \in \tilde{\mathbb{S}}$ such that $\ell(s_i \tilde{w}_1 s_i) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. By the proof of [6, Theorem 1.6],

$$\sharp X_{\tilde{w}}(\dot{x})(\mathbb{F}_q) = \sharp X_{\tilde{w}_1}(\dot{x})(\mathbb{F}_q) = (q-1)\sharp X_{s_i \tilde{w}_1}(\dot{x})(\mathbb{F}_q) + q\sharp X_{s_i \tilde{w}_1 s_i}(\dot{x})(\mathbb{F}_q).$$

Hence by inductive hypothesis,

$$\begin{aligned}
\sharp X_{\tilde{w}}(\dot{x})(\mathbb{F}_q) &= (q-1)\sharp X_{s_i\tilde{w}_1}(\dot{x})(\mathbb{F}_q) + q\sharp X_{s_i\tilde{w}_1s_i}(\dot{x})(\mathbb{F}_q) \\
&= (q-1)nq^{\frac{\ell(\tilde{w})-1}{2}}f_{s_i\tilde{w}_1, \emptyset} \mid_{v=\sqrt{q}} + qnq^{\frac{\ell(\tilde{w})-2}{2}}f_{s_i\tilde{w}_1s_i, \emptyset} \mid_{v=\sqrt{q}} \\
&= nq^{\frac{\ell(\tilde{w})}{2}}((\sqrt{q} - \sqrt{q}^{-1})f_{s_i\tilde{w}_1, \emptyset} \mid_{v=\sqrt{q}} + f_{s_i\tilde{w}_1s_i, \emptyset} \mid_{v=\sqrt{q}}) \\
&= nq^{\frac{\ell(\tilde{w})}{2}}f_{\tilde{w}, \emptyset} \mid_{v=\sqrt{q}}.
\end{aligned}$$

□

9. REDUCTION METHOD: VIA PARTIAL CONJUGATION

9.1. In this section, we investigate reduction method via “partial conjugation” introduced in [14] and use it to compare the dimension of $X_{\tilde{w}}(b)$ for various \tilde{w} in the same $W \times W$ -coset of \tilde{W} .

Notice that any $W \times W$ -coset of \tilde{W} contains a unique maximal element and this element is of the form w_0t^μ for some $\mu \in P_+$. An element in this double coset is of the form $xt^\mu y$ for $x \in W$ and $y \in {}^{I(\mu)}W$. Here $I(\mu) = \{i \in \mathbb{S}; \langle \mu, \alpha_i \rangle = 0\}$.

The main result of this section is

Theorem 9.1. *Let $\mu \in P_+$, $x \in W$ and $y \in {}^{I(\mu)}W$. Then for any $b \in G(L)$,*

$$\dim X_{xt^\mu y}(b) \leq \dim X_{w_0t^\mu}(b) - \ell(w_0) + \ell(x).$$

In particular, if $X_{xt^\mu y}(b) \neq \emptyset$, then $X_{w_0t^\mu}(b) \neq \emptyset$.

In the rest of this section, we’ll prove this Theorem. The proof is rather technical and relies heavily on a detailed analysis of partial conjugation.

9.2. We consider the “partial conjugation” action of W on \tilde{W} defined by $w \cdot_\delta w' = ww'\delta(w)^{-1}$ for $w \in W$ and $w' \in \tilde{W}$.

For $x \in {}^{\mathbb{S}}\tilde{W}$, set

$$I(x) = \max\{J \subseteq \mathbb{S}; \text{Ad}(x)\delta(J) = J\}.$$

This is well-defined as $\text{Ad}(x)\delta(J_1 \cup J_2) = J_1 \cup J_2$ if $\text{Ad}(x)\delta(J_i) = J_i$ for $i = 1, 2$.

By [14, Corollary 2.6], we have that

$$\tilde{W} = \sqcup_{x \in {}^{\mathbb{S}}\tilde{W}} W \cdot_\delta (W_{I(x)}x) = \sqcup_{x \in {}^{\mathbb{S}}\tilde{W}} W \cdot_\delta (xW_{\delta(I(x))}).$$

Moreover, we have the following result (see [14, Proposition 3.4]).

Theorem 9.2. *For any $\tilde{w} \in \tilde{W}$, there exists $x \in {}^{\mathbb{S}}\tilde{W}$ and $u \in W_{I(x)}$ such that*

$$\tilde{w} \xrightarrow{i_1}_\delta \cdots \xrightarrow{i_k}_\delta ux,$$

here $i_1, \dots, i_k \in \mathbb{S}$.

Now we apply this Theorem to the affine Deligne-Lusztig varieties.

Lemma 9.3. *Let $x, y \in W$ and $\mu \in P_+$ with $y \in {}^{I(\mu)}W$. Then there exists $w' \in {}^{I(\mu)}W$, such that*

$$\dim X_{xt^\mu y}(b) \leq \dim X_{t^\mu w'}(b) + \ell(x).$$

Proof. We prove the Lemma by induction on $\ell(x)$. Suppose the statement is true for all x' with $\ell(x') < \ell(x)$ but fails for x . Let $x = s_{i_1} \cdots s_{i_k}$ be a reduced expression of x . There are four different cases

- (1) $ys_{\delta(i_1)} < y$. In this case, $ys_{\delta(i_1)} \in {}^{I(\mu)}W$.
- (2) $ys_{\delta(i_1)} > y$ and $ys_{\delta(i_1)} \in {}^{I(\mu)}W$.
- (3) $ys_{\delta(i_1)} = s_{i_{k+1}}y$ for some $i_{k+1} \in I(\mu)$ and $\ell(s_{i_1}xs_{i_{k+1}}) = \ell(x) - 2$.
- (4) $ys_{\delta(i_1)} = s_{i_{k+1}}y$ for some $i_{k+1} \in I(\mu)$ and $\ell(s_{i_1}xs_{i_{k+1}}) = \ell(x)$.

If $ys_{\delta(i_1)} < y$, then $\ell(xt^\mu y) = \ell(s_{i_1}xt^\mu ys_{\delta(i_1)})$. By Proposition 4.2, $\dim X_{xt^\mu y}(b) = \dim X_{s_{i_1}xt^\mu ys_{\delta(i_1)}}(b)$. By induction hypothesis for $s_{i_1}x$, there exists $w' \in {}^{I(\mu)}W$ such that

$$\dim X_{xt^\mu y}(b) \leq \dim X_{t^\mu w'}(b) + \ell(s_{i_1}x) = \dim X_{t^\mu w'}(b) + \ell(x) - 1.$$

That contradicts our assumption on x .

If $ys_{\delta(i_1)} > y$ and $ys_{\delta(i_1)} \in {}^{I(\mu)}W$, then $\ell(xt^\mu y) = \ell(s_{i_1}xt^\mu ys_{\delta(i_1)}) + 2$. By Proposition 4.2,

$$\dim X_{xt^\mu y}(b) = \max\{\dim X_{s_{i_1}xt^\mu ys_{\delta(i_1)}}(b), \dim X_{s_{i_1}xt^\mu y}(b)\} + 1.$$

By induction hypothesis for $s_{i_1}x$, there exists $w' \in {}^{I(\mu)}W$ such that $\dim X_{xt^\mu y}(b) \leq \dim X_{t^\mu w'}(b) + \ell(s_{i_1}x) + 1 = \dim X_{t^\mu w'}(b) + \ell(x)$. That contradicts our assumption on x .

Therefore, $ys_{\delta(i_1)} = s_{i_{k+1}}y$ for some $i_{k+1} \in I(\mu)$. So $s_{i_1}xt^\mu ys_{\delta(i_1)} = s_{i_1}xs_{i_{k+1}}$. If $\ell(s_{i_1}xs_{i_{k+1}}) = \ell(x) - 2$, then $\ell(xt^\mu y) = \ell(s_{i_1}xs_{i_{k+1}}t^\mu y) + 2$. Hence

$$\dim X_{xt^\mu y}(b) = \max\{\dim X_{s_{i_1}xs_{i_{k+1}}t^\mu y}(b), \dim X_{s_{i_1}xt^\mu y}(b)\} + 1.$$

By induction hypothesis for $s_{i_1}x$, there exists $w' \in {}^{I(\mu)}W$ such that $\dim X_{xt^\mu y}(b) \leq \dim X_{t^\mu w'}(b) + \ell(x) - 1$. That contradicts our assumption on x .

Hence only case (4) can happen. Now apply the same argument to $s_{i_1}xs_{i_{k+1}} = s_{i_2} \cdots s_{i_{k+1}}$ instead of $x = s_{i_1} \cdots s_{i_k}$, we have that $ys_{\delta(i_2)} = s_{i_{k+2}}y$ for some $i_{k+2} \in I(\mu)$ and $\ell(s_{i_2} \cdots s_{i_{k+1}}) = \ell(s_{i_3} \cdots s_{i_{k+2}})$. Repeat the same procedure, one may define inductively $i_n \in I(\mu)$ for all $n > k$ by $s_{i_n} = ys_{\delta(i_{n-k})}y^{-1}$ and prove that $\ell(s_{i_{n-k+1}} \cdots s_{i_n}) = k$. In particular, $i_1, \dots, i_k \in I(t^\mu y)$. By Proposition 4.8, $\dim X_{xt^\mu y}(b) = \dim X_{t^\mu y}(b) + \ell(x)$. That also contradicts our assumption on x . \square

9.3. Now we recall Bédard's description of JW [1].

Let $\mathcal{T}(J)$ be the set of all sequences $(J_n, w_n)_{n \geq 1}$, where $J_n \subseteq \mathbb{S}$ and $w_n \in W$ such that

- (1) $J_1 = J$;
- (2) $J_n = J \cap \text{Ad}(w_{n-1})\delta(J_{n-1})$ for $n > 1$;
- (3) $w_n \in {}^JW^{\delta(J_n)}$ and $w_{n+1} \in w_n W_{\delta(J_n)}$ for all n .

Then $J_m = J_{m+1} = \dots$, $w_m = w_{m+1} = \dots$ and $\text{Ad}(w)\delta(J_m) = J_m$ for $m \gg 0$. Moreover $(J_n, w_n)_{n \geq 1} \mapsto w_m$ for $m \gg 0$ is a well-defined bijection between $\mathcal{T}(J)$ and JW .

Lemma 9.4. *Let $\mu \in P_+$ and $w \in {}^{I(\mu)}W$, then*

$$\dim X_{t^\mu w}(b) \leq \dim X_{w_0 t^\mu}(b) - \ell(w_0).$$

Proof. Let $J = I(\mu)$ and $(J_n, w_n)_{n \geq 1}$ be the element in $\mathcal{T}(J)$ that corresponds to $w \in {}^JW$. Then there exists $m \in \mathbb{N}$ such that $J_m = J_{m+1} = \dots$ and $w_m = w_{m+1} = \dots$. We prove by descending induction on $n \leq m$ that

$$(a) \quad \dim X_{t^\mu w}(b) \leq \dim X_{w_0^{J_n} t^\mu w_n}(b) - \ell(w_0^{J_n}).$$

Notice that $\text{Ad}(w)\delta(J_m) = J_m$. Thus $J_m \subseteq I(t^\mu w)$. By Proposition 4.8,

$$\dim X_{w_0^{J_m} t^\mu w}(b) = \dim X_{t^\mu w}(b) + \ell(w_0^{J_m}).$$

Thus (a) holds for $n = m$. Now assume that $n > 1$ and (a) holds for n , we'll prove that (a) also holds for $n - 1$.

Set $u = \delta^{-1}(w_n^{-1}w_{n-1})$. Then $w_{n-1} = w_n \delta(u)$. By definition of $\mathcal{T}(J)$, $u \in W_{J_{n-1}}$, $u^{-1} \in W^{J_n}$ and $\ell(w_n) = \ell(w_{n-1}) + \ell(u)$. Now $\ell(u^{-1}w_0^{J_n} t^\mu w_{n-1}) = \ell(w_0^{J_n} t^\mu w_n) + 2\ell(u)$. By Proposition 4.2,

$$(b) \quad \dim X_{w_0^{J_n} t^\mu w_n}(b) \leq \dim X_{u^{-1}w_0^{J_n} t^\mu w_{n-1}}(b) - \ell(u).$$

We prove that

$$(c) \text{ For any } u' \in W^{J_n} \text{ and } i \in J_{n-1} \text{ with } s_i u' < u', \dim X_{u' w_0^{J_n} t^\mu w_{n-1}}(b) \geq \dim X_{s_i u' w_0^{J_n} t^\mu w_{n-1}}(b) + 1.$$

Note that $w_{n-1} \in {}^{J_{n-1}}W^{\delta(J_{n-1})}$. Thus $\ell(w_{n-1} s_{\delta(i)}) = \ell(w_{n-1}) + 1$. There are two possibilities.

(i) $w_{n-1} s_{\delta(i)} \in {}^{J_{n-1}}W$.

Then

$$\begin{aligned} \ell(s_i u' w_0^{J_n} t^\mu w_{n-1} s_{\delta(i)}) &= \ell(s_i u' w_0^{J_n}) + \ell(t^\mu) - \ell(w_{n-1} s_{\delta(i)}) \\ &= \ell(u' w_0^{J_n}) - 1 + \ell(t^\mu) - \ell(w_{n-1}) - 1 \\ &= \ell(u' w_0^{J_n} t^\mu w_{n-1}) - 2. \end{aligned}$$

(ii) $w_{n-1} s_{\delta(i)} \notin {}^{J_{n-1}}W$.

Then $w_{n-1}s_{\delta(i)}w_{n-1}^{-1}$ is a simple reflection in $W_{J_{n-1}}$. By §9.3 (2), it is a simple reflection in W_{J_n} . Hence

$$\begin{aligned}
\ell(s_i u' w_0^{J_n} t^\mu w_{n-1} s_{\delta(i)}) &= \ell(s_i u' w_0^{J_n} w_{n-1} s_{\delta(i)} w_{n-1}^{-1} t^\mu w_{n-1}) \\
&= \ell(s_i u' w_0^{J_n} w_{n-1} s_{\delta(i)} w_{n-1}^{-1}) + \ell(t^\mu w_{n-1}) \\
&= \ell(s_i u') + \ell(w_0^{J_n} w_{n-1} s_{\delta(i)} w_{n-1}^{-1}) + \ell(t^\mu w_{n-1}) \\
&= \ell(u') + \ell(w_0^{J_n}) - 2 + \ell(t^\mu w_{n-1}) \\
&= \ell(u' w_0^{J_n} t^\mu w_{n-1}) - 2.
\end{aligned}$$

In either case, $\ell(s_i u' w_0^{J_n} t^\mu w_{n-1} s_{\delta(i)}) = \ell(u' w_0^{J_n} t^\mu w_{n-1}) - 2$. (c) follows from Proposition 4.2.

Now $u^{-1} \in W_{J_{n-1}} \cap W^{J_n}$. Then $w_0^{J_{n-1}} w_0^{J_n} = v u^{-1}$ for some $v \in W_{J_{n-1}}$ with $\ell(v u^{-1}) = \ell(v) + \ell(u)$. Let $v = s_{i_1} \cdots s_{i_k}$ be a reduced expression. Here $k = \ell(w_0^{J_{n-1}} w_0^{J_n}) - \ell(u)$. Then $s_{i_j} s_{i_{j+1}} \cdots s_{i_k} u^{-1} \in W^{J_n}$ for all j . By (c),

$$\begin{aligned}
\dim X_{u^{-1} w_0^{J_n} t^\mu w_{n-1}}(b) &\leq \dim X_{s_{i_k} u^{-1} w_0^{J_n} t^\mu w_{n-1}}(b) - 1 \\
&\leq \dim X_{s_{i_{k-1}} s_{i_k} u^{-1} w_0^{J_n} t^\mu w_{n-1}}(b) - 2 \\
&\leq \cdots \\
&\leq \dim X_{v u^{-1} w_0^{J_n} t^\mu w_{n-1}}(b) - k \\
&= \dim X_{w_0^{J_{n-1}} t^\mu w_{n-1}}(b) - \ell(w_0^{J_{n-1}} w_0^{J_n}) + \ell(u).
\end{aligned}$$

By (b) and induction hypothesis,

$$\begin{aligned}
\dim X_{t^\mu w}(b) &\leq \dim X_{w_0^{J_n} t^\mu w_n}(b) - \ell(w_0^{J_n}) \\
&\leq \dim X_{u^{-1} w_0^{J_n} t^\mu w_{n-1}}(b) - \ell(w_0^{J_n}) - \ell(u) \\
&\leq \dim X_{w_0^{J_{n-1}} t^\mu w_{n-1}}(b) - \ell(w_0^{J_{n-1}} w_0^{J_n}) - \ell(w_0^{J_n}) \\
&= \dim X_{w_0^{J_{n-1}} t^\mu w_{n-1}}(b) - \ell(w_0^{J_{n-1}}).
\end{aligned}$$

(a) is proved.

In particular, $\dim X_{t^\mu w}(b) \leq \dim X_{w_0^J t^\mu w_1}(b) - \ell(w_0^J)$. Here $w_1 \in {}^J W^{\delta(J)}$. By the same argument as we did for (b), we have that

$$\dim X_{w_0^J t^\mu w_1}(b) \leq \dim X_{\delta^{-1}(w_1) w_0^J t^\mu}(b) - \ell(w_1).$$

Similar to the proof of (c), we have that

(d) Let $u' \in W^J$ and $i \in \mathbb{S}$ with $s_i u' < u'$. Then $\dim X_{u' w_0^J t^\mu}(b) \geq \dim X_{s_i u' w_0^J t^\mu}(b) + 1$.

Let $w_0 w_0^J \delta^{-1}(w_1)^{-1} = s_{j_1} \cdots s_{j_l}$ be a reduced expression. Here $l = \ell(w_0) - \ell(w_0^J) - \ell(w_1)$. Then by (d), we have that

$$\begin{aligned} \dim X_{w_1 w_0^J t^\mu}(b) &\leq \dim X_{s_{i_l} w_1 w_0^J t^\mu}(b) - 1 \\ &\leq \dim X_{s_{i_{l-1}} s_{i_l} w_1 w_0^J t^\mu}(b) - 2 \\ &\leq \cdots \\ &\leq \dim X_{w_0 t^\mu}(b) - \ell(w_0) + \ell(w_0^J) + \ell(w_1). \end{aligned}$$

Now

$$\begin{aligned} \dim X_{t^\mu w}(b) &\leq \dim X_{w_0^J t^\mu w_1}(b) - \ell(w_0^J) \\ &\leq \dim X_{\delta^{-1}(w_1) w_0^J t^\mu}(b) - \ell(w_0^J) - \ell(w_1) \\ &\leq \dim X_{w_0 t^\mu}(b) - \ell(w_0). \end{aligned}$$

□

9.4. Now we prove Theorem 9.1. By Lemma 9.3, there exists $w \in {}^J W$ such that $\dim X_{xt^\mu y}(b) \leq \dim X_{t^\mu w}(b) + \ell(x)$. By Lemma 9.4, $\dim X_{t^\mu w}(b) \leq \dim X_{w_0 t^\mu}(b) - \ell(w_0)$. Hence

$$\dim X_{xt^\mu y}(b) \leq \dim X_{w_0 t^\mu}(b) - \ell(w_0) + \ell(x).$$

10. VIRTUAL DIMENSION

In this section, we discuss some applications of Theorem 9.1. First we compare the dimension of affine Deligne-Lusztig varieties in affine Grassmannian and affine flag.

Theorem 10.1. *For $\mu \in P_+$ and $b \in G(L)$,*

$$\dim X_\mu(b) = \dim X_{w_0 t^\mu}(b) - \ell(w_0).$$

Proof. Let $\pi : \mathcal{Fl} \rightarrow Gr$ be the projection. Then each fiber of π is isomorphic to $L^+ \mathcal{G}(\mathbf{k})/I$, which is of dimension $\ell(w_0)$. We have that $\pi^{-1} X_\mu(b) = \sqcup_{\tilde{w} \in W t^\mu W} X_{\tilde{w}}(b)$.

By Theorem 9.1, $\dim X_{\tilde{w}}(b) \leq \dim X_{w_0 t^\mu}(b)$ for all $\tilde{w} \in W t^\mu W$. Hence

$$\begin{aligned} \dim X_{w_0 t^\mu}(b) &= \dim(\sqcup_{\tilde{w} \in W t^\mu W} X_{\tilde{w}}(b)) = \dim \pi^{-1} X_\mu(b) \\ &= \dim X_\mu(b) + \ell(w_0). \end{aligned}$$

□

By combining Theorem 10.1 with Theorem 7.1, we have the following “converse to Mazur’s inequality” in the Iwahori case.

Corollary 10.2. *Let $J \subseteq \mathbb{S}$ with $\delta(J) = J$. Let $\mu \in P_+$ and $b \in M_J(L)$ be a basic element such that $\kappa_{M_J}(b) \in Y_J^+$. Then $X_{w_0 t^\mu}(b) \neq \emptyset$ if and only if $\kappa_{M_J}(b) \preceq_J \mu$.*

10.1. We define $\eta_\delta: \tilde{W} \rightarrow W$ as follows. If $\tilde{w} = xt^\mu y$ with $\mu \in P_+$, $x \in W$ and $y \in {}^{I(\mu)}W$, then we set $\eta_\delta(x) = \delta^{-1}(y)x$.

For $\tilde{w} \in \tilde{W}$ with $\kappa(x) = \kappa(b)$, we define the *virtual dimension*:

$$d_{\tilde{w}}(b) = \frac{1}{2}(\ell(\tilde{w}) + \ell(\eta_\delta(\tilde{w})) - \text{def}(b)) - \langle \bar{\nu}_b, \rho \rangle.$$

Here $\text{def}(b)$ is the defect of b . See [27].

Now we may compare dimension of affine Deligne-Lusztig variety with virtual dimension.

Theorem 10.3. *If $\delta = \text{id}$, then for any $b \in G(L)$ and $\tilde{w} \in \tilde{W}$ with $\kappa(\tilde{w}) = \kappa(b)$, we have that*

$$\dim X_{\tilde{w}}(b) \leq d_{\tilde{w}}(b).$$

Proof. As discussed in the proof of Theorem 7.1, $d_{\tilde{w}}(b) - \dim X_{\tilde{w}}(b)$ only depends on the combinatorial data $(\tilde{W}, \tilde{w}, f(b))$. Thus it suffices to consider the split case.

We may assume that $\tilde{w} = x\epsilon^\mu y$ with $x \in W$ and $\mu \in P_+$ and $y \in {}^{I(\mu)}W$. Let $\delta^{-1}(y) = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression. Then

$$\delta^{-1}(y)xt^\mu \xrightarrow{i_1}_\delta s_{i_1}\delta^{-1}(y)xt^\mu s_{\delta(i_1)} \xrightarrow{i_2}_\delta \cdots \xrightarrow{i_k}_\delta xt^\mu y.$$

By Proposition 4.2,

$$\dim X_{xt^\mu y}(b) \leq \dim X_{\delta^{-1}(y)xt^\mu}(b) + \frac{1}{2}(\ell(xt^\mu y) - \ell(\delta^{-1}(y)xt^\mu)).$$

By Theorem 9.1 and Theorem 10.1,

$$\begin{aligned} \dim X_{xt^\mu y}(b) &\leq \dim X_{w_0 t^\mu}(b) - \ell(w_0) + \frac{1}{2}(\ell(xt^\mu y) + \ell(\delta^{-1}(y)x)) - \langle \mu, \rho \rangle \\ &= \dim X_\mu(b) + \frac{1}{2}(\ell(xt^\mu y) + \ell(\delta^{-1}(y)x)) - \langle \mu, \rho \rangle. \end{aligned}$$

By [9] and [37], $\dim X_\mu(b) = \langle \mu - \bar{\nu}_b, \rho \rangle - \frac{1}{2}\text{def}(b)$. Therefore $\dim X_{\tilde{w}}(b) \leq d_{\tilde{w}}(b)$. \square

11. CLASS POLYNOMIAL: LOWEST TWO-SIDED CELL CASE

We have established an upper bound for the dimension of affine Deligne-Lusztig variety in the last section. In this section, we study in details of class polynomials for the lowest two-sided cell case. The main result is Theorem 11.4, which is a key step in establishing a lower bound for the dimension of affine Deligne-Lusztig variety.

11.1. Let $\tilde{w} \in \tilde{W}$, let $\text{supp}(\tilde{w})$ be the set of simple reflections of \tilde{S} that appears in some (or equivalently, any) reduced expression of \tilde{w} and $\text{supp}_\delta(\tilde{w}) = \cup_{i \in \mathbb{N}} \delta^i(\text{supp}(\tilde{w}))$.

By [16, Lemma 1], for any $\tilde{w}, \tilde{w}' \in \tilde{W}$, the set $\{uu'; u \leq \tilde{w}, u' \leq \tilde{w}'\}$ has a unique maximal element, which we denote by $\tilde{w} * \tilde{w}'$. Then $*$ is associative. Moreover, $\text{supp}(\tilde{w} * \tilde{w}') = \text{supp}(\tilde{w}) \cup \text{supp}(\tilde{w}')$ and $\text{supp}_\delta(\tilde{w} * \tilde{w}') = \text{supp}_\delta(\tilde{w}) \cup \text{supp}_\delta(\tilde{w}')$.

Let $A_+ = \mathbb{Z}_+[v - v^{-1}]$ and $\tilde{H}_+ = \sum_{\tilde{w} \in \tilde{W}} A_+ T_{\tilde{w}}$. Notice that $T_s^2 = (v - v^{-1})T_s + 1$ for any $s \in \tilde{\mathbb{S}}$. We have that

Lemma 11.1. *For any $i \in \tilde{\mathbb{S}}$, $\tilde{H}_+ T_{s_i} \subseteq \tilde{H}_+$.*

Proof. Let $\tilde{w} \in \tilde{W}$. Then

$$(a) \quad T_{\tilde{w}} T_{s_i} = \begin{cases} T_{\tilde{w}s_i}, & \text{if } \tilde{w}s_i > \tilde{w}; \\ T_{\tilde{w}} T_{s_i} = (v - v^{-1})T_{\tilde{w}} + T_{\tilde{w}s_i}, & \text{otherwise.} \end{cases}$$

In particular, $T_{\tilde{w}} T_{s_i} \in \tilde{H}_+$ and $A_+ T_{\tilde{w}} T_{s_i} \subseteq \tilde{H}_+$. \square

Lemma 11.2. *For any $x, y \in \tilde{W}$, we have that*

- (1) $T_x T_y \in T_{xy} + \tilde{H}_+$.
- (2) $T_x T_y \in (v - v^{-1})^{\ell(x) + \ell(y) - \ell(xy)} T_{x*y} + \tilde{H}_+$.

Proof. We argue by induction on $\ell(y)$. If $\ell(y) = 0$, then $x * y = xy$ and $T_x T_y = T_{xy}$. The statement is obvious.

Now assume that $\ell(y) > 0$ and the statement holds for all $y' \in \tilde{W}$ with $\ell(y') < \ell(y)$. Let $i \in \tilde{\mathbb{S}}$ with $ys_i < y$. Set $y' = ys_i$. Then $y = y' * s_i$. By induction hypothesis on y' ,

$$T_x T_y = T_x T_{y'} T_{s_i} \in (T_{xy'} + \tilde{H}_+) T_{s_i} \subseteq T_{xy'} T_{s_i} + \tilde{H}_+$$

and

$$\begin{aligned} T_x T_y &= T_x T_{y'} T_{s_i} \in ((v - v^{-1})^{\ell(x) + \ell(y') - \ell(xy')} T_{x*y'} + \tilde{H}_+) T_{s_i} \\ &\subseteq (v - v^{-1})^{\ell(x) + \ell(y') - \ell(xy')} T_{x*y'} T_{s_i} + \tilde{H}_+. \end{aligned}$$

By (a) of Lemma 11.1, $T_{xy'} T_{s_i} \in T_{xy's_i} + \tilde{H}_+$ and

$$T_{x*y'} T_{s_i} \in (v - v^{-1})^{\ell(x*y') + 1 - \ell((x*y') * s_i)} T_{(x*y') * s_i} + \tilde{H}_+.$$

Notice that $(x * y') * s_i = x * (y' * s_i) = x * y$. Hence $T_x T_y \in T_{xy'} T_{s_i} + \tilde{H}_+ = T_{xy} + \tilde{H}_+$ and

$$\begin{aligned} T_x T_y &\in (v - v^{-1})^{\ell(x) + \ell(y') - \ell(xy')} T_{x*y'} T_{s_i} + \tilde{H}_+ \\ &\subseteq (v - v^{-1})^{\ell(x) + \ell(y) - \ell(xy)} T_{x*y} + \tilde{H}_+. \end{aligned}$$

\square

As a consequence, we have that

Corollary 11.3. *For any $\tilde{w} \in \tilde{W}$, $T_{\tilde{w}} \tilde{H}_+ \subseteq \tilde{H}_+$ and $\tilde{H}_+ T_{\tilde{w}} \subseteq \tilde{H}_+$.*

11.2. We follow [36, 7.3]. Let (W, \mathbb{S}) be a Coxeter system and $\delta : W \rightarrow W$ which sends simple reflections to simple reflections. For each δ -orbit in \mathbb{S} we pick a simple reflection and let c be the product of the corresponding simple reflections (in any order). We call c a δ -twisted Coxeter element of W .

The main result we'll prove in this section is

Theorem 11.4. *Assume that G is quasi-simple. Let \tilde{W}' be the lowest two-sided cells of \tilde{W} . Let $\tilde{w} \in \tilde{W}' \cap \tau W_a$ for $\tau \in \Omega$. Let n be the number of δ -orbits on $\tilde{\mathbb{S}}$. Then there exists a maximal proper $\text{Ad}(\tau) \circ \delta$ -stable subset J of $\tilde{\mathbb{S}}$ and a $\text{Ad}(\tau) \circ \delta$ -twisted Coxeter element c of W_J such that*

$$T_{\tilde{w}} \in (v - v^{-1})^{\ell(\eta_{\delta}(\tilde{w})) - n} T_{\tau c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta}.$$

The proof relies on the following three Propositions.

Proposition 11.5. *Let $\tilde{w} = xt^{\mu}y$ with $\mu \in P_+$, $x \in W$ and $y \in {}^{I(\mu)}W$. If $\tilde{w} \in \tilde{W}'$ and $\text{supp}_{\delta}(\delta^{-1}(y)x) = \mathbb{S}$, then there exists $a \in W$ with $\text{supp}_{\delta}(a) = \mathbb{S}$ and $\gamma \in P_+$ such that*

$$T_{\tilde{w}} \in (v - v^{-1})^{\ell(\eta_{\delta}(\tilde{w})) - \ell(a)} T_{at\gamma} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta}.$$

Proof. Let $J = \{i \in \mathbb{S}; s_i y < y\}$. Since $y \in {}^{I(\mu)}W$, $J \cap I(\mu) = \emptyset$. Hence $\mu - \rho_J^{\vee} \in P_+$. Let $J' = I(\mu - \rho_J^{\vee})$. Then $\delta^{-1}(y)x = wz$ for some $w \in W^{J'}$ and $z \in W_{J'}$. Define $\gamma \in Y_+$ and $y' \in W^{I(\gamma)}$ by $\mu - \rho_J^{\vee} + w^{-1}\rho_{\delta^{-1}(J)}^{\vee} = y'\gamma$.

Set $\tilde{w}_1 = xz^{-1}t^{\mu - \rho_J^{\vee}}y'$ and $\tilde{w}_2 = (y')^{-1}zt^{\rho_J^{\vee}}y$. Then $\tilde{w} = \tilde{w}_1\tilde{w}_2$ and

$$\begin{aligned} \tilde{w}_2\delta(\tilde{w}_1) &= (y')^{-1}zt^{\rho_J^{\vee}}y\delta(\tilde{w}_1) = (y')^{-1}zt^{\rho_J^{\vee}}y\delta(xz^{-1})\delta(t^{\mu - \rho_J^{\vee}}y') \\ &= (y')^{-1}zt^{\rho_J^{\vee}}\delta(w)\delta(t^{\mu - \rho_J^{\vee}}y') = (y')^{-1}z\delta(wt^{\mu - \rho_J^{\vee} + w^{-1}\rho_{\delta^{-1}(J)}^{\vee}}y') \\ &= (y')^{-1}z\delta(wt^{y'\gamma}y') = (y')^{-1}z\delta(wy't^{\gamma}). \end{aligned}$$

By [8, §3.5], $\ell(\tilde{w}) = \ell(\tilde{w}_1) + \ell(\tilde{w}_2)$ and $\ell((y')^{-1}z) + \ell(wy') = \ell(\delta^{-1}(y)x)$. Set $a = ((y')^{-1}z)*\delta(wy')$. Then $\text{supp}_{\delta}(a) = \mathbb{S}$ since $\mathbb{S} = \text{supp}_{\delta}(\delta^{-1}(y)x) \subseteq \text{supp}_{\delta}(wy') \cup \text{supp}_{\delta}((y')^{-1}z)$. We have that

$$\begin{aligned} T_{\tilde{w}} &= T_{\tilde{w}_1}T_{\tilde{w}_2} \in T_{\tilde{w}_2}T_{\delta(\tilde{w}_1)} + [\tilde{H}, \tilde{H}]_{\delta} = T_{(y')^{-1}z}T_{t^{\rho_J^{\vee}}y}T_{\delta(\tilde{w}_1)} + [\tilde{H}, \tilde{H}]_{\delta} \\ &\subseteq T_{(y')^{-1}z}T_{\delta(wy't^{\gamma})} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta} = T_{(y')^{-1}z}T_{\delta(wy')}T_{t^{\gamma}} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta} \\ &\subseteq (v - v^{-1})^{\ell(\delta^{-1}(y)x) - \ell(a)} T_a T_{t^{\gamma}} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta} \\ &\subseteq (v - v^{-1})^{\ell(\delta^{-1}(y)x) - \ell(a)} T_{at\gamma} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta}. \end{aligned}$$

□

Proposition 11.6. *Assume that G is quasi-simple. Let $J \subseteq \mathbb{S}$ with $\delta(J) = J$ and $\tilde{w} = xt^{\mu}y$ such that $x \in W_J$ with $\text{supp}_{\delta}(v) = J$, y is a δ -twisted Coxeter element in $W_{\mathbb{S}-J}$, $\mu \neq 0$ and $t^{\mu}y \in {}^{\mathbb{S}}\tilde{W}$. Then there exists a δ -twisted Coxeter element c of W such that*

$$T_{\tilde{w}} \in (v - v^{-1})^{\ell(x) + \ell(y) - \ell(c)} T_{t^{\mu}c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_{\delta}.$$

Proof. We proceed by induction on $|J|$. Suppose that the statement is true for all $J' \subsetneq J$, but not true for J . We may also assume that the statement is true for all x' with $\text{supp}_{\delta}(x') = J$ and $\ell(x') < \ell(x)$, but not true for x .

Since G is quasi-simple and $\mu \neq 0$, there exists $i \in J$ such that $t^\mu y s_i \in {}^S\tilde{W}$. Let $J_1 = \{i \in J; t^\mu y s_i \notin {}^S\tilde{W}\}$. Then J_1 is a proper subset of J . For any $i \in J_1$, $t^\mu y s_i = s_j t^\mu y$ for some $j \in S$. Since $y \in W_{S-J}$, this is possible only if $j = i$ and j commutes with y .

We prove that

(a) $x \in W_{\delta^{-1}(J_1)}$.

We write x as ux_1 for $u \in W_{\delta^{-1}(J_1)}$ and $x_1 \in \delta^{-1}(J_1)W$. Then $T_{\tilde{w}} \equiv T_{x_1 t^\mu y} T_{\delta(u)} = T_{x_1} T_{\delta(u)} T_{t^\mu y} \pmod{[\tilde{H}, \tilde{H}]_\delta}$. Let $x' = x_1 * \delta(u)$. Notice that $\text{supp}_\delta(x) = \text{supp}_\delta(u) \cup \text{supp}_\delta(x_1) = \text{supp}_\delta(x_1) \cup \text{supp}_\delta(\delta(u)) = \text{supp}_\delta(x')$. Hence $\text{supp}(x') = J$. By Lemma 11.2 and Corollary 11.3,

$$T_{\tilde{w}} \in T_{x_1 t^\mu y} T_{\delta(u)} + [\tilde{H}, \tilde{H}]_\delta \subseteq (v - v^{-1})^{\ell(x) - \ell(x')} T_{x' t^\mu y} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta.$$

If $x \notin W_{\delta^{-1}(J_1)}$, then $x_1 \neq 1$ and there exists $i \in \delta^{-1}(J_1)$ with $s_i x_1 < x_1$. Then $s_i x' < x'$. Moreover, $\ell(y s_{\delta(i)}) = \ell(y) + 1$ and $t^\mu y s_{\delta(i)} \in {}^S\tilde{W}$. Hence

$$\begin{aligned} \ell(s_i x' t^\mu y s_{\delta(i)}) &= \ell(s_i x') + \ell(t^\mu) - \ell(y s_{\delta(i)}) = \ell(x') + \ell(t^\mu) - \ell(y) - 2 \\ &= \ell(x' t^\mu y) - 2. \end{aligned}$$

If $\text{supp}_\delta(s_i x') = J$, then

$$\begin{aligned} T_{x' t^\mu y} &= T_{s_i} T_{s_i x' t^\mu y} \in T_{s_i x' t^\mu y} T_{s_{\delta(i)}} + [\tilde{H}, \tilde{H}]_\delta \\ &\subseteq (v - v^{-1}) T_{s_i x' t^\mu y} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta. \end{aligned}$$

By induction hypothesis on $s_i x'$, we have that

$$T_{s_i x' t^\mu y} \in (v - v^{-1})^{\ell(x') - 1 + \ell(y) - \ell(c)} T_{t^\mu c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta$$

for some δ -twisted Coxeter element c of W .

Therefore $T_{\tilde{w}} \in (v - v^{-1})^{\ell(x) + \ell(y) - \ell(c)} T_{t^\mu c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta$. That is a contradiction.

If $\text{supp}_\delta(s_i x') \neq J$, then

$$\begin{aligned} T_{x' t^\mu y} &= T_{s_i} T_{s_i x' t^\mu y} \in T_{s_i x' t^\mu y} T_{s_{\delta(i)}} + [\tilde{H}, \tilde{H}]_\delta \\ &\subseteq T_{s_i x' t^\mu y s_{\delta(i)}} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta. \end{aligned}$$

By induction hypothesis on $\text{supp}_\delta(s_i x')$, we have that

$$T_{s_i x' t^\mu y s_{\delta(i)}} \in (v - v^{-1})^{\ell(x') + \ell(y) - \ell(c)} T_{t^\mu c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta$$

for some δ -twisted Coxeter element c of W .

Therefore $T_{\tilde{w}} \in (v - v^{-1})^{\ell(x) + \ell(y) - \ell(c)} T_{t^\mu c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta$. That is also a contradiction.

Now (a) is proved.

We have that $T_{\tilde{w}} \in T_{t^\mu y} T_{\delta(x)} + [\tilde{H}, \tilde{H}]_\delta = T_{\delta(x) t^\mu y} + [\tilde{H}, \tilde{H}]_\delta$. By the same argument for $\delta(x)$ instead of x , we have that $\delta(x) \in W_{\delta^{-1}(J_1)}$. Repeat the same procedure, $\delta^i(x) \in W_{\delta^{-1}(J_1)}$ for all i . Thus $\text{supp}_\delta(x) \subseteq \delta^{-1}(J_1) \subsetneq J$. That is again a contradiction. \square

Proposition 11.7. *Assume that G is quasi-simple. Let $\tau \in \Omega$. Then there exists a maximal $\text{Ad}(\tau) \circ \delta$ -stable proper subset J of $\tilde{\mathbb{S}}$ and a $\text{Ad}(\tau) \circ \delta$ -twisted Coxeter element c of W_J such that τc is a minimal length element in its δ -conjugacy class of \tilde{W} and $t^\mu w \rightarrow_\delta \tau c$ for any δ -Coxeter element w of W and $\mu \in P$ with $\kappa(t^\mu) = \kappa(\tau)$.*

This is the main result of [21]. A partial result for some classical groups was proved in [17].

11.3. Now we prove Theorem 11.4. By Proposition 11.5, there exists $a \in W$ with $\text{supp}_\delta(a) = S$ and $\lambda \in P_+$ such that

$$(a) \quad T_{\tilde{w}} \in (v - v^{-1})^{\ell(\eta_\delta(\tilde{w})) - \ell(a)} T_{at^\gamma} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta.$$

By Proposition 11.6, there exists a δ -twisted Coxeter element w of W such that

$$(b) \quad T_{at^\gamma} \in (v - v^{-1})^{\ell(a) - n} T_{t^\gamma w} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta.$$

Now by Proposition 11.7, there exists maximal $\text{Ad}(\tau) \circ \delta$ -stable proper subset J of $\tilde{\mathbb{S}}$ and a $\text{Ad}(\tau) \circ \delta$ -twisted Coxeter element c of W_J such that $t^\gamma a \rightarrow_\delta \tau c$. Thus by §2.3,

$$(c) \quad T_{t^\gamma w} \in T_{\tau c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta.$$

The theorem then follows from (a), (b) and (c).

12. CONJECTURE OF GHKR

Now we give a lower bound of $\dim X_{\tilde{w}}(b)$.

Theorem 12.1. *Assume that G is quasi-simple. Let $\tilde{w} \in \tilde{W}'$ and $b \in G(L)$ is a basic element with $\text{supp}_\delta(\eta_\delta(\tilde{w})) = \mathbb{S}$ and $\kappa(\tilde{w}) = \kappa(b)$. Then $\dim X_{\tilde{w}}(b) \geq d_{\tilde{w}}(b)$.*

Remark. It is proved in [10], [8] and [11] that if $\tilde{w} \in \tilde{W}'$ and $b \in G(L)$ is a basic element with $\kappa(\tilde{w}) = \kappa(b)$, then $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $\text{supp}_\delta(\eta_\delta(\tilde{w})) = \mathbb{S}$.

Proof. By Theorem 11.4, there exists a maximal proper $\text{Ad}(\tau) \circ \delta$ -stable subset J of $\tilde{\mathbb{S}}$ and a $\text{Ad}(\tau) \circ \delta$ -twisted Coxeter element c of W_J such that

$$T_{\tilde{w}} \in (v - v^{-1})^{\ell(\eta_\delta(\tilde{w})) - n} T_{\tau c} + \tilde{H}_+ + [\tilde{H}, \tilde{H}]_\delta.$$

Let \mathcal{O} be the δ -conjugacy class of \tilde{W} that contains τc . Then $f_{\tilde{w}, \mathcal{O}} \in (v - v^{-1})^{\ell(\eta_\delta(\tilde{w})) - n} + A_+$. In particular, $\deg f_{\tilde{w}, \mathcal{O}} \geq \ell(\eta_\delta(\tilde{w})) - n$. Here n is the number of δ -orbits on \mathbb{S} .

By Theorem 6.1,

$$\begin{aligned} \dim X_{\tilde{w}}(b) &\geq \frac{1}{2}(\ell(\tilde{w}) + \ell(c) + \deg f_{\tilde{w}, \mathcal{O}}) \\ &\geq \frac{1}{2}(\ell(\tilde{w}) + \ell(\eta_\delta(\tilde{w})) + \ell(c) - n). \end{aligned}$$

By the definition of defect, $\text{def}(b) = \text{def}(\dot{\tau}) = n - \ell(c)$. Therefore $\dim X_{\tilde{w}}(b) \geq d_{\tilde{w}}(b)$. \square

By combining Theorem 12.1 and Theorem 10.1, we have that

Corollary 12.2. *If G is quasi-simple and $\delta = \text{id}$. Let $\tilde{w} \in \tilde{W}'$ and $b \in G(L)$ be a basic element with $\text{supp}(\eta(\tilde{w})) = \mathbb{S}$ and $\kappa(\tilde{w}) = \kappa(b)$. Then $\dim X_{\tilde{w}}(b) = d_{\tilde{w}}(b)$.*

Remark. The split case was first conjectured in [10]. A weaker result for some split classical groups was proved in [8].

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